Logic in Action

Chapter 2: Propositional Logic

http://www.logicinaction.org/

Example (1)

In a restaurant, your Father has ordered Fish, your Mother ordered Vegetarian, and you ordered Meat. Out of the kitchen comes some new person carrying the three plates. What will happen?

Example (2)

Three guests are sitting at a table. The waitress asks: **"Does everyone want coffee"**. The first guest says: **"I don't know"**. The second guest now says: **"I don't know"**. Then the third guest says: **"No, not everyone wants coffee"**. The waitress comes back and gives the right people their coffees. Assuming that at the beginning each guest only knows about himself, which was the waitress reasoning? Who gets coffee and who does not?

Example (3)

1	·	·
·	·	2
·	·	·

Example (3)

1	·	**3**
·	·	2
·	·	·

Example (3)

1	**2**	3
.	.	2
.	.	**1**

Example (3)

1	2	3
3	·	2
·	**3**	1

Example (3)

1	2	3
3	**1**	2
2	3	1

Example (4)

If you take the medication, you will get better.

You are taking the medication.

? ―――――――――――――――――――――

So, you will get better.

Example (4)

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So, you took the medication.

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So, you will get better.

If you take the medication, you will get better.
You are getting better.

✗ ———————————————

So, you took the medication.

Example (5)

If you take the medication, you will get better.

But you are not taking the medication.

? ______________________________

So, you will not get better.

Example (5)

If you take the medication, you will get better.

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So, you will not get better.

Example (5)

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If you take the medication, you will get better.
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So, you will not get better.

?

If you take the medication, you will get better.
But you are not getting better.

So, you have not taken the medication.

Example (5)

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If you take the medication, you will get better.
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Valid inference

$$\frac{A_1, \ldots, A_n}{C}$$

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$$\frac{A_1, \ldots, A_n}{C}$$

An inference is **valid** if and only if **every** time **all** the premises are true, the conclusion is also true.

What a valid inference tells us?

Suppose the following inference is valid

$$\frac{A_1, \ldots, A_n}{C}$$

Then

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What a valid inference tells us?

Suppose the following inference is valid

$$\frac{A_1, \ldots, A_n}{C}$$

Then

1. if **all** the premises $A_1, \ldots, A_n$ are true, so is the conclusion C.
2. if the conclusion C is false, **at least one** premise A_i is false.

Looking for patterns (1)

Two valid inferences:

If you take the medication, then you will get better.
You are taking the medication.

So, you will get better.

If you jump from a 4th floor, then you will fly.
You jump from a 4th floor.

So, you will fly.

Looking for patterns (1)

Two valid inferences:

If **you take the medication**, then **you will get better**.
You are taking the medication.

So, **you will get better**.

If **you jump from a 4th floor**, then **you will fly**.
You jump from a 4th floor.

So, **you will fly**.

Looking for patterns (1)

Two valid inferences:

If ***A***, then ***B***.
A.

So, ***B***.

If ***E***, then ***F***.
E.

So, ***F***.

Looking for patterns (2)

Another valid inference:

If you take the medication, then you will get better.

You are not getting better.

So, you are not taking the medication.

Looking for patterns (2)

Another valid inference:

If **you take the medication**, then **you will get better**.

You are not getting better.

So, **you are not taking the medication**.

Looking for patterns (2)

Another valid inference:

If ***A***, then ***B***.
not ***B***.

So, not ***A***.

Looking for patterns (3)

And yet another:

An integer x is even or odd.
If x is even, then $x + x$ is even.
If x is odd, then $x + x$ is even.

So, $x + x$ is even.

Looking for patterns (3)

And yet another:

An integer x is **even** or **odd**.

If x **is even**, then $x + x$ **is even**.

If x **is odd**, then $x + x$ **is even**.

So, $x + x$ **is even**.

Looking for patterns (3)

And yet another:

An integer x is $\boldsymbol{A_1}$ or $\boldsymbol{A_2}$.
If x is $\boldsymbol{A_1}$, then $\boldsymbol{C}$.
If x is $\boldsymbol{A_2}$, then $\boldsymbol{C}$.

So, $\boldsymbol{C}$.

Looking for patterns (3)

And yet another:

An integer x is $\boldsymbol{A_1}$ or $\boldsymbol{A_2}$.
If x is $\boldsymbol{A_1}$, then $\boldsymbol{C}$.
If x is $\boldsymbol{A_2}$, then $\boldsymbol{C}$.

So, $\boldsymbol{C}$.

Can you think of others?

The main question

How can we recognize valid inference patterns?

Example (6)

A1 At least one of them is guilty.

A2 Not all of them are guilty.

A3 If Mrs White is guilty, then Colonel Mustard helped her (he is guilty too).

A4 If Miss Scarlet is innocent then so is Colonel Mustard.

Example (6)

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There is a situation in which $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$ and $\boldsymbol{A_4}$ are all true, but **"Colonel Mustard is guilty"** is false (there is a **counter-example**).

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2. Operators to build more statements:

"**not** … "	becomes	$\neg \ldots$
"… **and** …"	becomes	$\ldots \wedge \ldots$
"… **or** …"	becomes	$\ldots \vee \ldots$
"**if** … **then**"	becomes	$\ldots \rightarrow \ldots$
" … **if and only if** …"	becomes	$\ldots \leftrightarrow \ldots$

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In practice, we will avoid parenthesis if they are not necessary.

Constructing formulas

The construction of a formula can be seen as building a **tree**.

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$$((\neg p \lor q) \to r)$$

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|

$$\rightarrow$$

Constructing formulas

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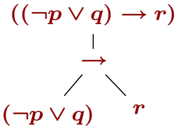

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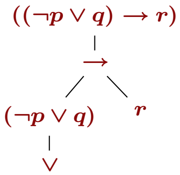

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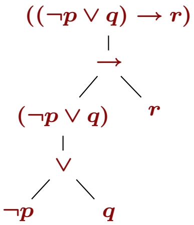

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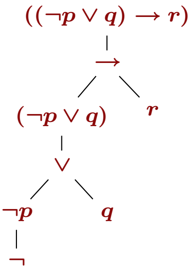

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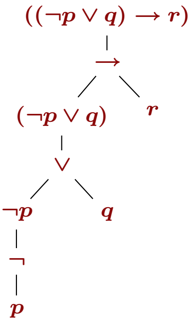

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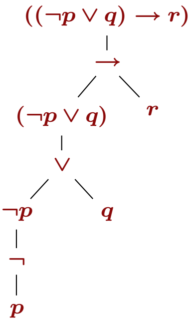

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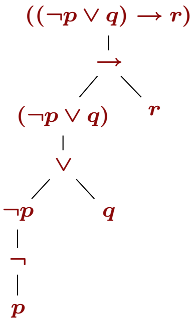

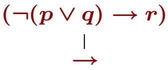

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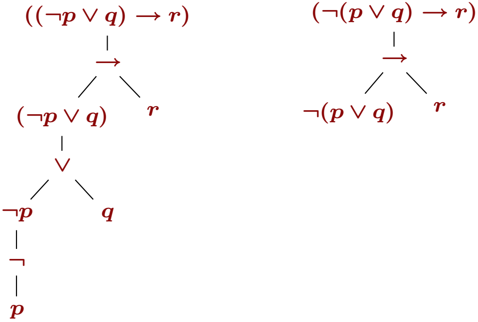

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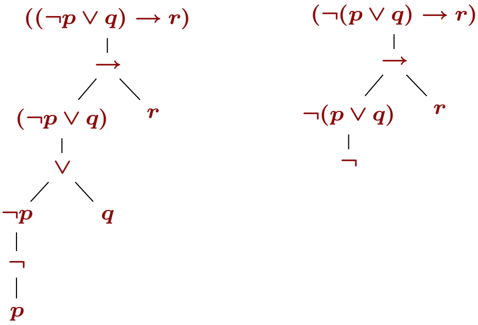

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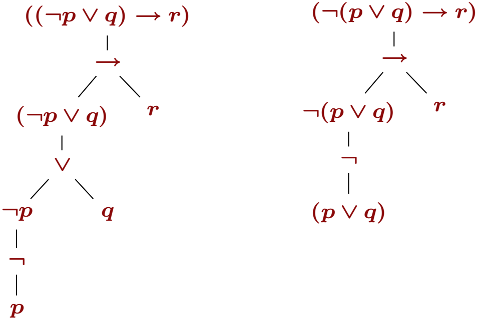

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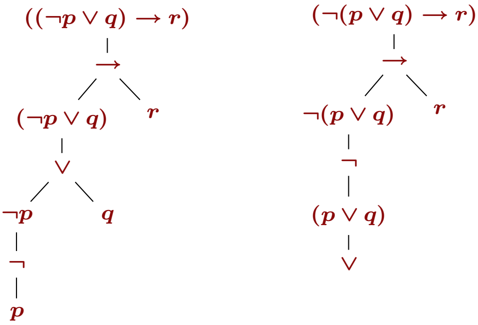

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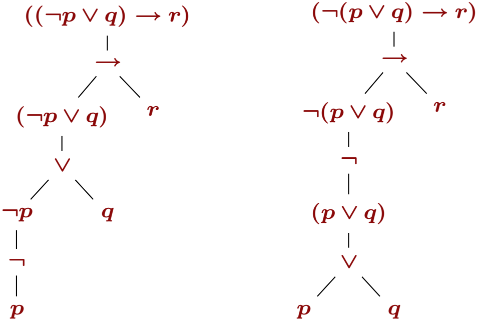

Evaluating formulas

How do we know if a given formula φ is **true** or **false**?

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- We need the **truth-values** of the basic propositions p, q, r, ... that appear in φ.

Evaluating formulas

How do we know if a given formula φ is **true** or **false**?

- We need the **truth-values** of the basic propositions $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, ... that appear in φ.
- We need to know the **meaning** of $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

Behaviour of the connectives (1)

Use 1 for **true**, and 0 for **false**.

For **negation** $\neg$

φ	$\neg\varphi$

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For **negation** $\neg$

φ	$\neg\varphi$
1	
0	

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Use 1 for **true**, and 0 for **false**.

For **negation** $\neg$

φ	$\neg\varphi$
1	**0**
0	

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φ	$\neg\varphi$
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or, in a shorter format:

$\neg$	φ

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φ	$\neg\varphi$
1	**0**
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or, in a shorter format:

$\neg$	φ
	1
	0

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φ	$\neg\varphi$
1	**0**
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or, in a shorter format:

$\neg$	φ
0	1
	0

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φ	$\neg\varphi$
1	**0**
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or, in a shorter format:

$\neg$	φ
0	1
1	0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1		1
1		0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1		0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ
1		1
1		0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ
1	**1**	1
1		0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ
1	**1**	1
1	**1**	0
0		1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ
1	**1**	1
1	**1**	0
0	**1**	1
0		0

Behaviour of the connectives (2)

For **conjunction** $\wedge$

φ	$\wedge$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**0**	0

For **disjunction** $\vee$

φ	$\vee$	ψ
1	**1**	1
1	**1**	0
0	**1**	1
0	**0**	0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1		1
1		0
0		1
0		0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1		0
0		1
0		0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0		1
0		0

Behaviour of the connectives (3)

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φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0		0

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φ	$\leftrightarrow$	ψ
1	**1**	1
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0	**0**	1
0	**1**	0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ
1		1
1		0
0		1
0		0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ
1	**1**	1
1		0
0		1
0		0

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For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ
1	**1**	1
1	**0**	0
0		1
0		0

Behaviour of the connectives (3)

For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ
1	**1**	1
1	**0**	0
0	**1**	1
0		0

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For **equivalence** $\leftrightarrow$

φ	$\leftrightarrow$	ψ
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

For **implication** $\rightarrow$

φ	$\rightarrow$	ψ
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

Valuations

Valuation. Let $\mathtt{P} = \{p, q, r, \ldots\}$ be a set of atomic propositions. A **valuation** V from $\mathtt{P}$ to $\{0, 1\}$ assigns to each element of $\mathtt{P}$ a unique truth-value.

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$V_1(p) = 1$	$V_1(q) = 1$

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$V_1(p) = 1$	$V_1(q) = 1$
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$V_3(p) = 0$	$V_3(q) = 1$

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$V_2(p) = 1$	$V_2(q) = 0$
$V_3(p) = 0$	$V_3(q) = 1$
$V_4(p) = 0$	$V_4(q) = 0$

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$V_2(p) = 1$	$V_2(q) = 0$
$V_3(p) = 0$	$V_3(q) = 1$
$V_4(p) = 0$	$V_4(q) = 0$

How many for $\mathtt{P} = \{p\}$? How many for $\mathtt{P} = \{p, q, r\}$?

Evaluating formulas in one situation

$(\neg \quad p) \quad \wedge \quad q$

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q	
V:		0		1	

Evaluating formulas in one situation

	($\neg$	p)	$\wedge$	q
V:	1	0		1

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$$V \models (\neg p) \wedge q$$

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

$(p \quad \wedge \quad (p \quad \rightarrow \quad q)) \quad \rightarrow \quad q$

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	1	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1		1		0		0

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$$\boxed{V \models (\neg p) \wedge q}$$

	$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
V:	1		1	0	0		0

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0		0

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	$(\neg$	$p)$	$\wedge$	q
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	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

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	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$$\boxed{V \models (\neg p) \wedge q}$$

	$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
V:	1	0	1	0	0	**1**	0

$$\boxed{V \models (p \wedge (p \to q)) \to q}$$

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \rightarrow q)) \rightarrow q$

$\neg \quad \neg \quad p$

Evaluating formulas in one situation

	($\neg$	p)	$\wedge$	q
V:	1	0	**1**	1

$$V \models (\neg p) \wedge q$$

	(p	$\wedge$	(p	$\to$	q))	$\to$	q
V:	1	0	1	0	0	**1**	0

$$V \models (p \wedge (p \to q)) \to q$$

	$\neg$	$\neg$	p
V:			0

Evaluating formulas in one situation

	$($\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \rightarrow q)) \rightarrow q$

	$\neg$	$\neg$	p
V:		1	0

Evaluating formulas in one situation

	($\neg$	p)	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	(p	$\wedge$	(p	$\rightarrow$	q))	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \rightarrow q)) \rightarrow q$

	$\neg$	$\neg$	p
V:	**0**	1	0

Evaluating formulas in one situation

	$($¬	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \to q)) \to q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

Evaluating formulas in one situation

	($\neg$	p)	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	(p	$\wedge$	(p	$\rightarrow$	q))	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \rightarrow q)) \rightarrow q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

$(p \;\rightarrow\; q) \;\vee\; (q \;\rightarrow\; p)$

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \to q)) \to q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

	$(p$	$\to$	$q)$	$\vee$	$(q$	$\to$	$p)$
V:	0		1		1		0

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \to q)) \to q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

	$(p$	$\to$	$q)$	$\vee$	$(q$	$\to$	$p)$
V:	0	1	1		1		0

Evaluating formulas in one situation

	($\neg$	p)	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	(p	$\wedge$	(p	$\to$	q))	$\to$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \to q)) \to q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

	(p	$\to$	q)	$\vee$	(q	$\to$	p)
V:	0	1	1		1	0	0

Evaluating formulas in one situation

	$($¬	$p)$	$\wedge$	q
V:	1	0	**1**	1

$$V \models (\neg p) \wedge q$$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$$V \models (p \wedge (p \rightarrow q)) \rightarrow q$$

	$\neg$	$\neg$	p
V:	**0**	1	0

$$V \not\models \neg\neg p$$

	$(p$	$\rightarrow$	$q)$	$\vee$	$(q$	$\rightarrow$	$p)$
V:	0	1	1	**1**	1	0	0

Evaluating formulas in one situation

	$(\neg$	$p)$	$\wedge$	q
V:	1	0	**1**	1

$V \models (\neg p) \wedge q$

	$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
V:	1	0	1	0	0	**1**	0

$V \models (p \wedge (p \rightarrow q)) \rightarrow q$

	$\neg$	$\neg$	p
V:	**0**	1	0

$V \not\models \neg\neg p$

	$(p$	$\rightarrow$	$q)$	$\vee$	$(q$	$\rightarrow$	$p)$
V:	0	1	1	**1**	1	0	0

$V \models (p \rightarrow q) \vee (q \rightarrow p)$

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
1		1		1		1
1		1		0		0
0		0		1		1
0		0		0		0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
1		1	1	1		1
1		1	0	0		0
0		0	1	1		1
0		0	1	0		0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
1	1	1	1	1		1
1	0	1	0	0		0
0	0	0	1	1		1
0	0	0	1	0		0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
1	1	1	1	1	**1**	1
1	0	1	0	0	**1**	0
0	0	0	1	1	**1**	1
0	0	0	1	0	**1**	0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
1	1	1	1	1	**1**	1
1	0	1	0	0	**1**	0
0	0	0	1	1	**1**	1
0	0	0	1	0	**1**	0

$\neg$	$\neg$	p

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
1	1	1	1	1	**1**	1
1	0	1	0	0	**1**	0
0	0	0	1	1	**1**	1
0	0	0	1	0	**1**	0

$\neg$	$\neg$	p
		1
		0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\to$	$q))$	$\to$	q
1	1	1	1	1	**1**	1
1	0	1	0	0	**1**	0
0	0	0	1	1	**1**	1
0	0	0	1	0	**1**	0

$\neg$	$\neg$	p
	0	1
	1	0

Evaluating formulas in all possible situations

$(p$	$\wedge$	$(p$	$\rightarrow$	$q))$	$\rightarrow$	q
1	1	1	1	1	**1**	1
1	0	1	0	0	**1**	0
0	0	0	1	1	**1**	1
0	0	0	1	0	**1**	0

$\neg$	$\neg$	p
1	0	1
0	1	0

Classification of formulas according to their behaviour

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Classification of formulas according to their behaviour

- Those that are never true (**contradiction**):

$$p \wedge (\neg p), \ldots$$

- Those that can be true (**satisfiable**):

$$(\neg p) \vee q, \ldots$$

- Those that are always true (**valid**, **tautology**):

$$(p \wedge (p \rightarrow q)) \rightarrow q, \ldots$$

If the formula φ is valid, we write $\models \varphi$

Valid inference

$$\text{Inference:} \quad \frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

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Valid inference. An inference is **valid** if and only if **every** time (**every** situation) in which **all** premises $\varphi_1, \ldots, \varphi_n$ are true, ψ is also true.

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We also say ψ is a **logical consequence** of $\varphi_1, \ldots, \varphi_n$.

We will write $\varphi_1, \ldots, \varphi_n \models \psi$

Examples

Our previous patterns:

p	$(p \rightarrow q)$		q

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p	$(p$	$\rightarrow$	$q)$	q
1	1	**1**	1	**1**
1	1	**0**	0	**0**
0	0	**1**	1	**1**
0	0	**1**	0	**0**

Examples

Our previous patterns:

	p	$(p$	$\to$	$q)$	q
$\longrightarrow$	**1**	1	**1**	1	**1**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
	0	0	**1**	0	**0**

Examples

Our previous patterns:

	p	$(p$	$\rightarrow$	$q)$	q
$\rightarrow$	**1**	1	**1**	1	**1**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
	0	0	**1**	0	**0**

$\neg q$	$(p$	$\rightarrow$	$q)$	$\neg p$

Examples

Our previous patterns:

	p	$(p$	$\rightarrow$	$q)$	q
$\rightarrow$	**1**	1	**1**	1	**1**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
	0	0	**1**	0	**0**

$\neg q$	$(p$	$\rightarrow$	$q)$	$\neg p$
0	1	**1**	1	**0**
1	1	**0**	0	**0**
0	0	**1**	1	**1**
1	0	**1**	0	**1**

Examples

Our previous patterns:

	p	$(p$	$\rightarrow$	$q)$	q
$\longrightarrow$	**1**	1	**1**	1	**1**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
	0	0	**1**	0	**0**

	$\neg q$	$(p$	$\rightarrow$	$q)$	$\neg p$
	0	1	**1**	1	**0**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
$\longrightarrow$	**1**	0	**1**	0	**1**

Examples

Our previous patterns:

	p	$(p$	$\rightarrow$	$q)$	q
$\rightarrow$	**1**	1	**1**	1	**1**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
	0	0	**1**	0	**0**

	$\neg q$	$(p$	$\rightarrow$	$q)$	$\neg p$
	0	1	**1**	1	**0**
	1	1	**0**	0	**0**
	0	0	**1**	1	**1**
$\rightarrow$	**1**	0	**1**	0	**1**

What about the others?

Further definitions

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- Two formulas φ and ψ are **logically equivalent** ($\varphi \equiv \psi$) if and only if $\varphi \models \psi$ and $\psi \models \varphi$.

Further definitions

- Two formulas φ and ψ are **logically equivalent** ($\varphi \equiv \psi$) if and only if $\varphi \models \psi$ and $\psi \models \varphi$.
- A set of formulas X is **satisfiable** if and only if **there is one valuation** that makes **every** formula in X true.

Symbolic inference

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- A **proof** is a finite sequence of formulas where each formula is either an *axiom* or else it has been infered from previous formulas by using an **inference rule**.

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- A formula is a **theorem** if it occurs in a proof.
- A set of axioms and rules is called an **axiom system** or an **axiomatization**.
  - An axiom system is **sound** for a logic if every theorem is valid in the logic.
  - An axiom system is **complete** if every valid formula of the logic is a theorems.

Proof system

The following axiom system is sound and complete for propositional logic:

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2. $((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$.

Proof system

The following axiom system is sound and complete for propositional logic:

1. $(\varphi \to (\psi \to \varphi))$.
2. $((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$.
3. $((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$.

Proof system

The following axiom system is sound and complete for propositional logic:

1. $(\varphi \to (\psi \to \varphi))$.
2. $((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$.
3. $((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$.
4. **Modus ponens (MP)**: from φ and $\varphi \to \psi$, infer ψ.

Example

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1. $p \rightarrow ((q \rightarrow p) \rightarrow p)$ Instance of axiom 1

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2. $(p \rightarrow ((q \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p))$ Instance of axiom 2

Example

1.	$p \to ((q \to p) \to p)$	Instance of axiom 1
2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	Instance of axiom 2
3.	$(p \to (q \to p)) \to (p \to p)$	MP from steps 1 and 2

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2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	Instance of axiom 2
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4.	$(p \to (q \to p)$	Instance of axiom 1

Example

1.	$p \rightarrow ((q \rightarrow p) \rightarrow p)$	Instance of axiom 1
2.	$(p \rightarrow ((q \rightarrow p) \rightarrow p)) \rightarrow ((p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p))$	Instance of axiom 2
3.	$(p \rightarrow (q \rightarrow p)) \rightarrow (p \rightarrow p)$	MP from steps 1 and 2
4.	$(p \rightarrow (q \rightarrow p)$	Instance of axiom 1
5.	$p \rightarrow p$	MP from steps 4 and 3

Example

1.	$p \to ((q \to p) \to p)$	Instance of axiom 1
2.	$(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$	Instance of axiom 2
3.	$(p \to (q \to p)) \to (p \to p)$	MP from steps 1 and 2
4.	$(p \to (q \to p))$	Instance of axiom 1
5.	$p \to p$	MP from steps 4 and 3

Hence, $p \to p$ is a theorem.

Example

Mrs White is guilty.	w
Miss Scarlet is guilty.	s
Colonel Mustard is guilty.	m

Example

Mrs White is guilty.	w
Miss Scarlet is guilty.	s
Colonel Mustard is guilty.	m

- At least one of them is guilty.
- Not all of them are guilty.
- If Mrs White is guilty, then Colonel Mustard helped her.
- If Miss Scarlet is innocent then so is Colonel Mustard.

Example

Mrs White is guilty.	w
Miss Scarlet is guilty.	s
Colonel Mustard is guilty.	m

- At least one of them is guilty. $w \vee s \vee m$
- Not all of them are guilty. $\neg(w \wedge s \wedge m)$
- If Mrs White is guilty, then Colonel Mustard helped her. $w \rightarrow m$
- If Miss Scarlet is innocent then so is Colonel Mustard. $\neg s \rightarrow \neg m$

The questions

Define

$$\Phi := \{w \lor s \lor m, \neg(w \land s \land m), w \to m, \neg s \to \neg m\}$$

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Define

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- $\Phi \models s$?
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- $\Phi \models m$?

The questions

Define

$$\Phi := \{w \vee s \vee m, \neg(w \wedge s \wedge m), w \rightarrow m, \neg s \rightarrow \neg m\}$$

- $\Phi \models s$?
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- $\Phi \models m$?
- $\Phi \models \neg m$?

Updates

Which are the possibilities?

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$$\{\ \} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w, s\} \quad \{w, m\} \quad \{s, m\} \quad \{w, s, m\}$$

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$$\{\ \} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w, s\} \quad \{w, m\} \quad \{s, m\} \quad \{w, s, m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

Updates

Which are the possibilities?

$$\{\ \} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w, s\} \quad \{w, m\} \quad \{s, m\} \quad \{w, s, m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

$$\{\ \} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w, s\} \quad \{w, m\} \quad \{s, m\} \quad \{w, s, m\}$$

Updates

Which are the possibilities?

$$\{\,\} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

$$\{\,\} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

Updates

Which are the possibilities?

$$\{\,\} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

$$\sout{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

$$\sout{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \sout{\{w,s,m\}}$$

Updates

Which are the possibilities?

$$\{\,\} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

$$\cancel{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

$$\cancel{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \cancel{\{w,s,m\}}$$

Remove cases where the **third** premise $w \rightarrow m$ is false:

Updates

Which are the possibilities?

$\{\,\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ $\{w, s, m\}$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

~~$\{\,\}$~~ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ $\{w, s, m\}$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

~~$\{\,\}$~~ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ ~~$\{w, s, m\}$~~

Remove cases where the **third** premise $w \rightarrow m$ is false:

~~$\{\,\}$~~ ~~$\{w\}$~~ $\{s\}$ $\{m\}$ ~~$\{w, s\}$~~ $\{w, m\}$ $\{s, m\}$ ~~$\{w, s, m\}$~~

Updates

Which are the possibilities?

$$\{\,\} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

$$\cancel{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \{w,s,m\}$$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

$$\cancel{\{\,\}} \quad \{w\} \quad \{s\} \quad \{m\} \quad \{w,s\} \quad \{w,m\} \quad \{s,m\} \quad \cancel{\{w,s,m\}}$$

Remove cases where the **third** premise $w \longrightarrow m$ is false:

$$\cancel{\{\,\}} \quad \cancel{\{w\}} \quad \{s\} \quad \{m\} \quad \cancel{\{w,s\}} \quad \{w,m\} \quad \{s,m\} \quad \cancel{\{w,s,m\}}$$

Remove cases where the **fourth** premise $\neg s \longrightarrow \neg m$ is false:

Updates

Which are the possibilities?

$\{\,\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w,s\}$ $\{w,m\}$ $\{s,m\}$ $\{w,s,m\}$

Remove cases where the **first** premise $w \vee s \vee m$ is false:

~~$\{\,\}$~~ $\{w\}$ $\{s\}$ $\{m\}$ $\{w,s\}$ $\{w,m\}$ $\{s,m\}$ $\{w,s,m\}$

Remove cases where the **second** premise $\neg(w \wedge s \wedge m)$ is false:

~~$\{\,\}$~~ $\{w\}$ $\{s\}$ $\{m\}$ $\{w,s\}$ $\{w,m\}$ $\{s,m\}$ ~~$\{w,s,m\}$~~

Remove cases where the **third** premise $w \rightarrow m$ is false:

~~$\{\,\}$~~ ~~$\{w\}$~~ $\{s\}$ $\{m\}$ ~~$\{w,s\}$~~ $\{w,m\}$ $\{s,m\}$ ~~$\{w,s,m\}$~~

Remove cases where the **fourth** premise $\neg s \rightarrow \neg m$ is false:

~~$\{\,\}$~~ ~~$\{w\}$~~ $\{s\}$ ~~$\{m\}$~~ ~~$\{w,s\}$~~ ~~$\{w,m\}$~~ $\{s,m\}$ ~~$\{w,s,m\}$~~

Updates

Only the following possibilities make **all** premises in $\boldsymbol{\Phi}$ true:

~~$\{\}$~~ ~~$\{w\}$~~ $\{\boldsymbol{s}\}$ ~~$\{m\}$~~ ~~$\{w, s\}$~~ ~~$\{w, m\}$~~ $\{\boldsymbol{s}, \boldsymbol{m}\}$ ~~$\{w, s, m\}$~~

Updates

Only the following possibilities make **all** premises in $\boldsymbol{\Phi}$ true:

$\{\}$ $\{w\}$ $\{\boldsymbol{s}\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{\boldsymbol{s}, \boldsymbol{m}\}$ $\{w, s, m\}$

- $\boldsymbol{\Phi} \models \boldsymbol{s}$?

Updates

Only the following possibilities make **all** premises in $\mathbf{\Phi}$ true:

$\{\}$ $\{w\}$ $\{\boldsymbol{s}\}$ $\{m\}$ $\{w,s\}$ $\{w,m\}$ $\{\boldsymbol{s},\boldsymbol{m}\}$ $\{w,s,m\}$

- $\mathbf{\Phi} \models \boldsymbol{s}$? **Yes!**

Updates

Only the following possibilities make **all** premises in Φ true:

$\{\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ $\{w, s, m\}$

- $\Phi \models s$? **Yes!**
- $\Phi \models \neg w$?

Updates

Only the following possibilities make **all** premises in $\mathbf{\Phi}$ true:

$\{\}$ $\{w\}$ $\{\boldsymbol{s}\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{\boldsymbol{s}, \boldsymbol{m}\}$ $\{w, s, m\}$

- $\mathbf{\Phi} \models \boldsymbol{s}$? **Yes!**
- $\mathbf{\Phi} \models \neg \boldsymbol{w}$? **Yes!**

Updates

Only the following possibilities make **all** premises in Φ true:

$\{\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ $\{w, s, m\}$

- $\Phi \models s$? **Yes!**
- $\Phi \models \neg w$? **Yes!**
- $\Phi \models m$?

Updates

Only the following possibilities make **all** premises in Φ true:

~~$\{\}$~~ ~~$\{w\}$~~ $\{s\}$ ~~$\{m\}$~~ ~~$\{w, s\}$~~ ~~$\{w, m\}$~~ $\{s, m\}$ ~~$\{w, s, m\}$~~

- $\Phi \models s$? **Yes!**
- $\Phi \models \neg w$? **Yes!**
- $\Phi \models m$? **No!**

Updates

Only the following possibilities make **all** premises in Φ true:

$\{\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w, s\}$ $\{w, m\}$ $\{s, m\}$ $\{w, s, m\}$

- $\Phi \models s$? **Yes!**
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Only the following possibilities make **all** premises in Φ true:

$\{\}$ $\{w\}$ $\{s\}$ $\{m\}$ $\{w,s\}$ $\{w,m\}$ $\{s,m\}$ $\{w,s,m\}$

- $\Phi \models s$? **Yes!**
- $\Phi \models \neg w$? **Yes!**
- $\Phi \models m$? **No!**
- $\Phi \models \neg m$? **No!**

Do we need all that we have?

Decide whether the following formulas are logically equivalent:

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- $\varphi \wedge \psi$ and $\neg(\neg\varphi \vee \neg\psi)$.

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Decide whether the following formulas are logically equivalent:

- $\varphi \wedge \psi$ and $\neg(\neg\varphi \vee \neg\psi)$.
- $\varphi \rightarrow \psi$ and $\neg\varphi \vee \psi$.

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Decide whether the following formulas are logically equivalent:

- $\varphi \wedge \psi$ and $\neg(\neg\varphi \vee \neg\psi)$.
- $\varphi \rightarrow \psi$ and $\neg\varphi \vee \psi$.
- $\varphi \leftrightarrow \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Do we need all that we have?

Decide whether the following formulas are logically equivalent:

- $\varphi \land \psi$ and $\neg(\neg\varphi \lor \neg\psi)$.
- $\varphi \rightarrow \psi$ and $\neg\varphi \lor \psi$.
- $\varphi \leftrightarrow \psi$ and $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$.

- What does this tell us?

Do we need all that we have?

Decide whether the following formulas are logically equivalent:

- $\varphi \wedge \psi$ and $\neg(\neg\varphi \vee \neg\psi)$.
- $\varphi \rightarrow \psi$ and $\neg\varphi \vee \psi$.
- $\varphi \leftrightarrow \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

- What does this tell us?
- Can you find other set of operators strong enough to define the rest of them?

Do we have all that we need? (1)

Consider a single atomic proposition $\boldsymbol{p}$.

$\boldsymbol{p}$	

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Consider a single atomic proposition $\boldsymbol{p}$.

$\boldsymbol{p}$	
1	
0	

Do we have all that we need? (1)

Consider a single atomic proposition $\boldsymbol{p}$.

$\boldsymbol{p}$	φ_1
1	1
0	1

Do we have all that we need? (1)

Consider a single atomic proposition p.

p	φ_1	φ_2
1	1	1
0	1	0

Do we have all that we need? (1)

Consider a single atomic proposition p.

p	φ_1	φ_2	φ_3
1	1	1	0
0	1	0	1

Do we have all that we need? (1)

Consider a single atomic proposition p.

p	φ_1	φ_2	φ_3	φ_4
1	1	1	0	0
0	1	0	1	0

Do we have all that we need? (1)

Consider a single atomic proposition p.

p	φ_1	φ_2	φ_3	φ_4
1	1	1	0	0
0	1	0	1	0

- Can we define φ_1, φ_2, φ_3 and φ_4 in our setting?

Do we have all that we need? (1)

Consider a single atomic proposition p.

p	φ_1	φ_2	φ_3	φ_4
1	1	1	0	0
0	1	0	1	0

- Can we define φ_1, φ_2, φ_3 and φ_4 in our setting?
- Can we define each φ_i by using only p and our five connectives $\neg$, $\wedge$, $\vee$, $\longrightarrow$ and $\leftrightarrow$?

Do we have all that we need? (2)

Consider two atomic propositions p and q.

p q	

Do we have all that we need? (2)

Consider two atomic propositions p and q.

p	q	
1	1	
1	0	
0	1	
0	0	

Do we have all that we need? (2)

Consider two atomic propositions p and q.

p	q	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

Do we have all that we need? (2)

Consider two atomic propositions p and q.

p	q	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

- Can we define each φ_i by using only p, q and our five connectives $\neg$, $\wedge$, $\vee$, $\longrightarrow$ and $\leftrightarrow$?

Do we have all that we need? (3)

Consider three atomic propositions p, q and r.

p q r	

Do we have all that we need? (3)

Consider three atomic propositions p, q and r.

p	q	r	
1	1	1	
1	1	0	
1	0	1	
1	0	0	
0	1	1	
0	1	0	
0	0	1	
0	0	0	

Do we have all that we need? (3)

Consider three atomic propositions $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.

p	q	r	$\cdots$
1	1	1	
1	1	0	
1	0	1	
1	0	0	$\cdots$
0	1	1	
0	1	0	
0	0	1	
0	0	0	

Do we have all that we need? (3)

Consider three atomic propositions p, q and r.

p	q	r	$\cdots$
1	1	1	
1	1	0	
1	0	1	
1	0	0	
0	1	1	$\cdots$
0	1	0	
0	0	1	
0	0	0	

- Can we define each φ_i by using only p, q, r and our five connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$?