Logic in Action

Chapter 4: The World according to Predicate Logic

http://www.logicinaction.org/

Looking for further structure

Statement	Propositional translation
John reads	
John walks	

Looking for further structure

Statement	Propositional translation
John reads	p
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Statement	**Propositional translation**
John reads	$\boldsymbol{p}$
John walks	$\boldsymbol{q}$

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But the fact that both statements talk about "John" is lost.

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With a language including predicates . . .

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The language of **predicate logic** allow us

1. to talk about objects, their properties and their relations with other objects, and
2. to make use of **universal** and **existential** quantification.

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$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$$

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5. **Quantifiers**

$\forall x$ ("**for all** x") and $\exists x$ ("**there exists an** x")

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All A **are** B $\forall x(Ax \to Bx)$

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$\forall x(Ax \to Bx)$

$\exists x(Ax \land Bx)$

Example: syllogistic statements

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$\forall x(Ax \to Bx)$

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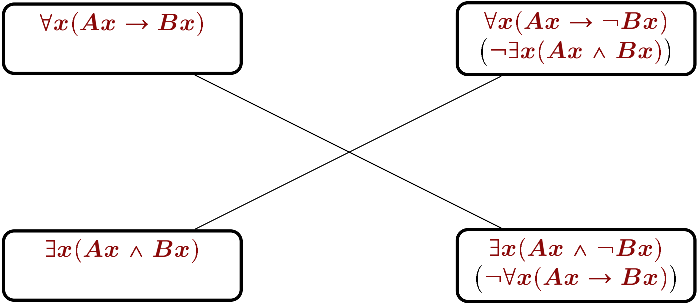

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- We can talk about **relations** between two or more objects:

John sees Mary:
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Everyone sees someone:
Someone sees everyone:
Everyone is seen by someone:
Someone is seen by everyone:

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Examples

Lxy	–	x loves y
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Bx	–	x is a boy

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Every girl who loves all boys does not love every girl	$\forall x\Big((Gx \wedge \underline{\varphi(x)}) \to \underline{\psi(x)}\Big)$

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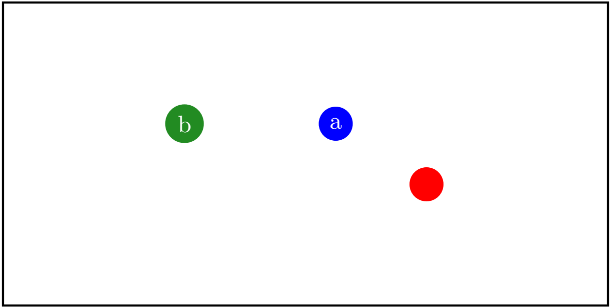
Evaluating predicate logic formulas (1)

Colors (***R***ed, ***G***reen, ***B***lue, ***P***urple) and shapes (***S***quare, ***C***ircle).



- Ba
- $\exists x Sx \vee Cb$
- $Ra \rightarrow Sb$
- $Ba \wedge Gb$
- $\neg Sa$
- $Ra \rightarrow \exists x Sx$

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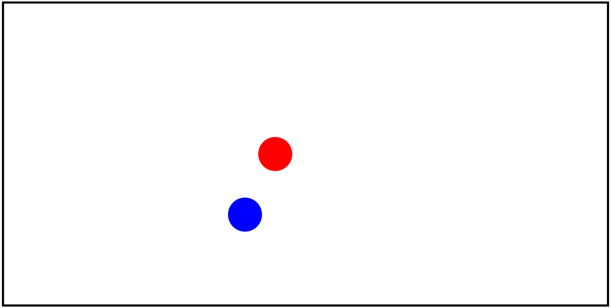

- $\exists x Rx$
- $\forall x(Rx \to Cx)$
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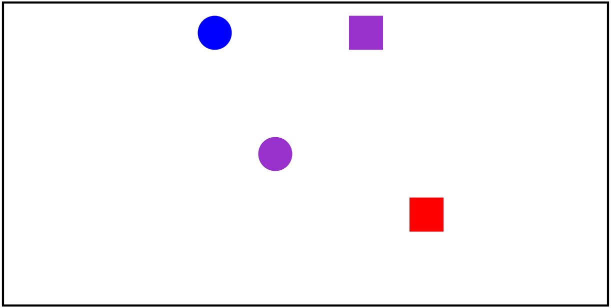
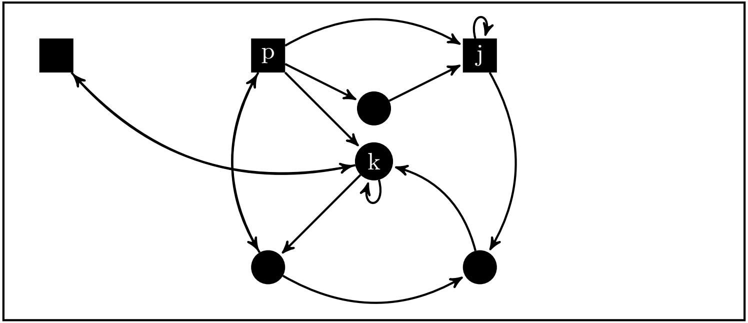


- $\exists x(Rx \land Cx)$
- $\forall x(Cx \lor Sx)$
- $\exists x Gx \lor \exists x Cx$
- $\exists x Rx \land \exists x Cx$
- $\forall x Cx \lor \forall x Sx$
- $\exists x(Gx \lor Cx)$

Evaluating predicate logic formulas (2)

■: boy ●: girl ● → ■: ● loves ■



- $Ljk \rightarrow Lkj$
- $\neg(Ljk \wedge Lkj)$
- $\forall x(Bx \rightarrow Lxk)$
- $\forall x((Bx \vee Gx) \rightarrow \neg Lxp)$
- $Ljk \wedge Lkj$
- $(Ljk \wedge Lpk) \rightarrow (\neg Lpj \wedge \neg Lkj)$
- $\neg\forall x(Gx \rightarrow Lxx)$
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   - If φ and ψ are formulas, then the following are formulas:
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   - If φ is a formula and x is a variable, the following are formulas:

     $$\forall x \varphi, \ \exists x \varphi$$

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- $\exists y \forall x (Lyx)$

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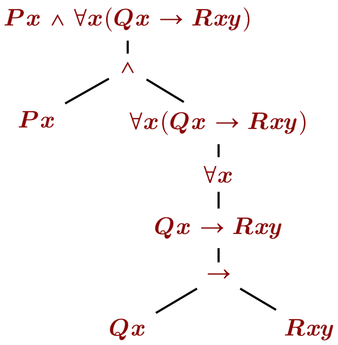
- **Bound variable.** An occurrence of a variable x is **bound** in a formula φ if there is a quantifier in φ that binds it.
- **Free variable.** An occurrence of a variable x is **free** in a formula φ if no quantifier in φ binds it.

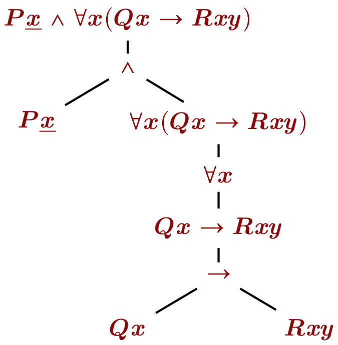
Example

$$Px \wedge \forall x(Qx \to Rxy)$$

Example


$Px \wedge \forall x(Qx \to Rxy)$
$\wedge$
Px
$\forall x(Qx \to Rxy)$
$\forall x$
$Qx \to Rxy$
$\to$
Qx
Rxy

Example



- **<u>This</u>** occurrence of x is **not bound** by any quantifier (is **free**).

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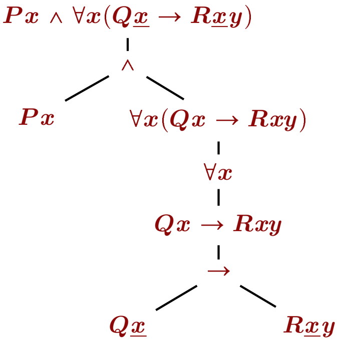

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- <u>**These**</u> occurrences of x are **bound** (by $\forall x$).

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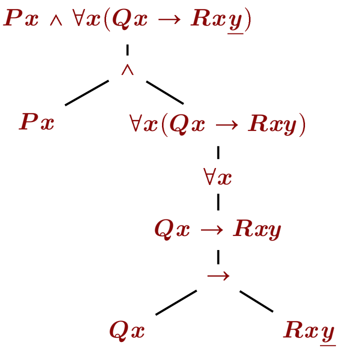

- This occurrence of x is **not bound** by any quantifier (is **free**).
- These occurrences of x are **bound** (by $\forall x$).
- **This** occurrence of y is **not bound** by any quantifier (is **free**).

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$$\text{For a } \textbf{constant}\text{:} \qquad (c)^y_t := c$$

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Examples:

$$(a)^x_c := a$$
$$(x)^y_a := x$$
$$(z)^z_y := y$$

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$$\begin{aligned}
(P t_1 \cdots t_n)^y_t &:= P (t_1)^y_t \cdots (t_n)^y_t \\
(\neg\varphi)^y_t &:= \neg(\varphi)^y_t \\
(\varphi \land \psi)^y_t &:= (\varphi)^y_t \land (\psi)^y_t \\
(\varphi \lor \psi)^y_t &:= (\varphi)^y_t \lor (\psi)^y_t \\
(\varphi \rightarrow \psi)^y_t &:= (\varphi)^y_t \rightarrow (\psi)^y_t \\
(\varphi \leftrightarrow \psi)^y_t &:= (\varphi)^y_t \leftrightarrow (\psi)^y_t
\end{aligned}$$

$$\begin{cases} (\forall x\varphi)^y_t := \forall x(\varphi)^y_t \\ (\forall y\varphi)^y_t := \forall y\varphi \end{cases}$$

$$\begin{cases} (\exists x\varphi)^y_t := \exists x(\varphi)^y_t \\ (\exists y\varphi)^y_t := \exists y\varphi \end{cases}$$

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For a constant a, sometimes we will abbreviate object $I(a)$ as a_I.

For a predicate P, sometimes we will abbreviate relation $I(P)$ as P_I.

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The **variable assignment** $g_{[x:=d]}$ differs from g only in the value of x, given now by d.

Assigning values to terms

Given a model $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle$, the value of every term $\boldsymbol{t}$, $[\![\boldsymbol{t}]\!]^{\boldsymbol{I}}_{\boldsymbol{g}}$, is defined as follows.

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For a **constant** $\boldsymbol{a}$: $[\![a]\!]^{\boldsymbol{I}}_{\boldsymbol{g}} := \boldsymbol{I}(\boldsymbol{a})$

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$$\langle D, I, g\rangle \models P t_1 \cdots t_n \quad \text{iff} \quad \left([\![t_1]\!]^I_g, \ldots, [\![t_n]\!]^I_g\right) \in I(P)$$

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$\langle D, I, g\rangle \models \varphi \vee \psi$	iff	$\langle D, I, g\rangle \models \varphi$ or $\langle D, I, g\rangle \models \psi$
$\langle D, I, g\rangle \models \varphi \to \psi$	iff	$\langle D, I, g\rangle \models \varphi$ implies $\langle D, I, g\rangle \models \psi$

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\langle D, I, g\rangle \models \neg\varphi & \text{iff} & \text{it is not the case that } \langle D, I, g\rangle \models \varphi \\
\langle D, I, g\rangle \models \varphi \wedge \psi & \text{iff} & \langle D, I, g\rangle \models \varphi \text{ and } \langle D, I, g\rangle \models \psi \\
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\langle D, I, g\rangle \models \varphi \to \psi & \text{iff} & \langle D, I, g\rangle \models \varphi \text{ implies } \langle D, I, g\rangle \models \psi \\
\langle D, I, g\rangle \models \varphi \leftrightarrow \psi & \text{iff} & \langle D, I, g\rangle \models \varphi \text{ if and only if } \langle D, I, g\rangle \models \psi
\end{array}$$

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$\langle D, I, g\rangle \models \forall x\varphi$	iff	for every $d \in D$ we have $\langle D, I, g_{[x:=d]}\rangle \models \varphi$

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$\langle D, I, g\rangle \models P t_1 \cdots t_n$	iff	$(\llbracket t_1\rrbracket^I_g, \ldots, \llbracket t_n\rrbracket^I_g) \in I(P)$
$\langle D, I, g\rangle \models \neg\varphi$	iff	it is not the case that $\langle D, I, g\rangle \models \varphi$
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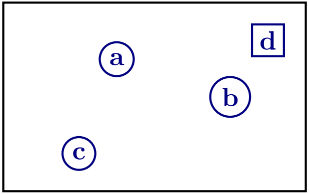
$\langle D, I, g\rangle \models \forall x\varphi$	iff	for every $d \in D$ we have $\langle D, I, g_{[x:=d]}\rangle \models \varphi$
$\langle D, I, g\rangle \models \exists x\varphi$	iff	there is a $d \in D$ such that $\langle D, I, g_{[x:=d]}\rangle \models \varphi$

Example

Shapes (***S***quare, ***C***ircle).



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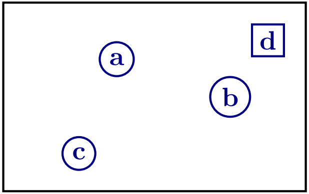

$D := \{ⓐ, ⓑ, ⓒ, \boxed{d}\}$

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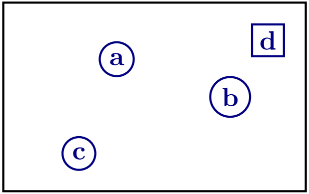

D := {ⓐ, ⓑ, ⓒ, d}

$I(a)$:= ⓐ

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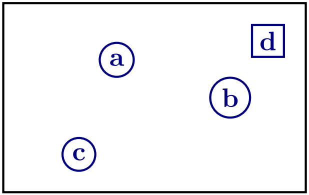

$\boldsymbol{D} := \{ⓐ, ⓑ, ⓒ, \boxed{d}\}$

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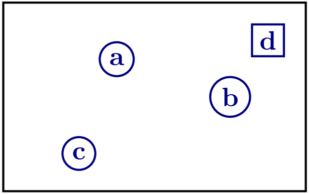

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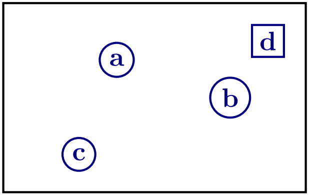

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$\langle D, I, g \rangle \models Ca$ iff

$\langle D, I, g \rangle \models Sx$ iff

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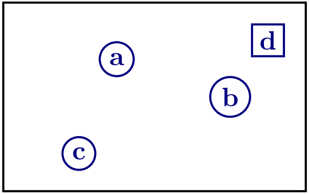

$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$

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$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Ca$ iff $[\![a]\!]_g^{\boldsymbol{I}} \in \boldsymbol{I}(C)$

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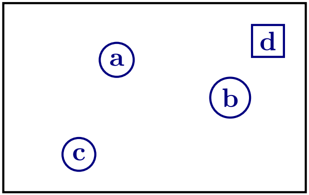

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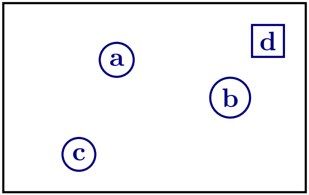

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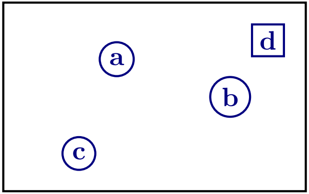

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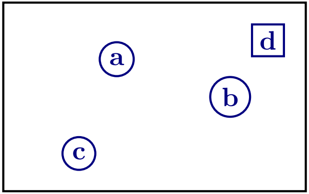

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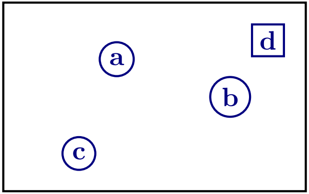
Shapes (***S***quare, ***C***ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(a) := \textcircled{a} \qquad \boldsymbol{I}(S) := \{\boxed{d}\}$$

$$\boldsymbol{I}(b) := \textcircled{b} \qquad \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(c) := \textcircled{c} \qquad \boldsymbol{g}(x) := \textcircled{b}$$

$$\boldsymbol{I}(d) := \boxed{d} \qquad \boldsymbol{g}(y) := \textcircled{a}$$

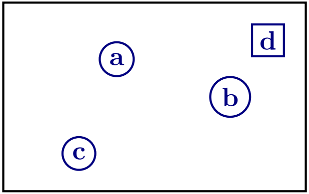
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $[\![x]\!]^{\boldsymbol{I}}_{\boldsymbol{g}} \in \boldsymbol{I}(S)$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$

$\boldsymbol{I}(a) := \textcircled{a}$	$\boldsymbol{I}(S) := \{\boxed{d}\}$
$\boldsymbol{I}(b) := \textcircled{b}$	$\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$
$\boldsymbol{I}(c) := \textcircled{c}$	$g(x) := \textcircled{b}$
$\boldsymbol{I}(d) := \boxed{d}$	$g(y) := \textcircled{a}$

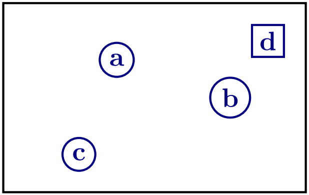
$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Sx$ iff $g(x) \in \boldsymbol{I}(S)$

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models \exists x Sx$ iff

Example

Shapes (***S***quare, ***C***ircle).



$D := \{ⓐ, ⓑ, ⓒ, \boxed{d}\}$

$I(a) := ⓐ$ $\qquad I(S) := \{\boxed{d}\}$

$I(b) := ⓑ$ $\qquad I(C) := \{ⓐ, ⓑ, ⓒ\}$

$I(c) := ⓒ$ $\qquad g(x) := ⓑ$

$I(d) := \boxed{d}$ $\qquad g(y) := ⓐ$

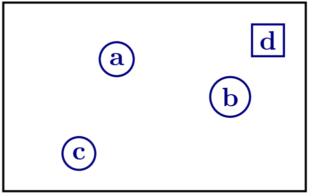
$\langle D, I, g \rangle \models Ca$ iff $ⓐ \in \{ⓐ, ⓑ, ⓒ\}$ ✓

$\langle D, I, g \rangle \models Sx$ iff $ⓑ \in I(S)$

$\langle D, I, g \rangle \models \exists x Sx$ iff

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(a) := \textcircled{a} \qquad \boldsymbol{I}(S) := \{\boxed{d}\}$$

$$\boldsymbol{I}(b) := \textcircled{b} \qquad \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(c) := \textcircled{c} \qquad \boldsymbol{g}(x) := \textcircled{b}$$

$$\boldsymbol{I}(d) := \boxed{d} \qquad \boldsymbol{g}(y) := \textcircled{a}$$

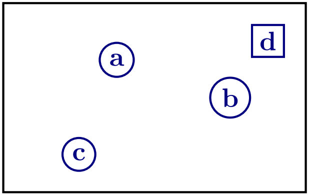
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Ca}$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Sx}$ iff $\textcircled{b} \in \{\boxed{d}\}$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists \boldsymbol{x} \boldsymbol{Sx}$ iff

Example

Shapes (***S***quare, ***C***ircle).



$$D := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$I(a) := \textcircled{a} \qquad I(S) := \{\boxed{d}\}$$

$$I(b) := \textcircled{b} \qquad I(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$I(c) := \textcircled{c} \qquad g(x) := \textcircled{b}$$

$$I(d) := \boxed{d} \qquad g(y) := \textcircled{a}$$

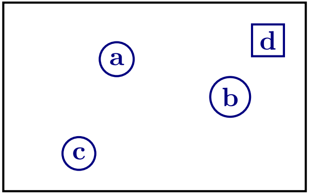
$$\langle D, I, g \rangle \models Ca \quad \text{iff} \quad \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \ \checkmark$$

$$\langle D, I, g \rangle \models Sx \quad \text{iff} \quad \textcircled{b} \in \{\boxed{d}\} \ ✗$$

$$\langle D, I, g \rangle \models \exists x Sx \quad \text{iff}$$

Example

Shapes (***S***quare, ***C***ircle).



$D := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$

$I(a) := \textcircled{a}$ $\qquad I(S) := \{\boxed{d}\}$

$I(b) := \textcircled{b}$ $\qquad I(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$

$I(c) := \textcircled{c}$ $\qquad g(x) := \textcircled{b}$

$I(d) := \boxed{d}$ $\qquad g(y) := \textcircled{a}$

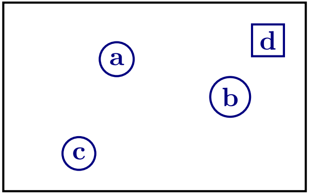
$\langle D, I, g\rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

$\langle D, I, g\rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle D, I, g\rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]}\rangle \models Sx$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(a) := \textcircled{a} \qquad \boldsymbol{I}(S) := \{\boxed{d}\}$$

$$\boldsymbol{I}(b) := \textcircled{b} \qquad \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(c) := \textcircled{c} \qquad \boldsymbol{g}(x) := \textcircled{b}$$

$$\boldsymbol{I}(d) := \boxed{d} \qquad \boldsymbol{g}(y) := \textcircled{a}$$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

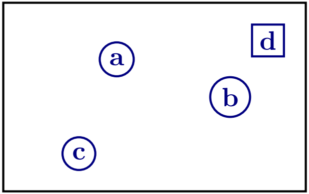
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=o]} \rangle \models Sx$

$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=\textcircled{a}]} \rangle \models Sx$$

Example

Shapes (***S***quare, ***C***ircle).



$$D := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$I(a) := \textcircled{a} \qquad I(S) := \{\boxed{d}\}$$

$$I(b) := \textcircled{b} \qquad I(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$I(c) := \textcircled{c} \qquad g(x) := \textcircled{b}$$

$$I(d) := \boxed{d} \qquad g(y) := \textcircled{a}$$

$\langle D, I, g\rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

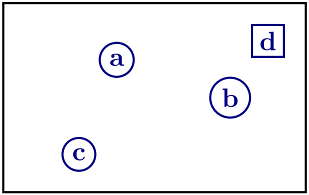
$\langle D, I, g\rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle D, I, g\rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]}\rangle \models Sx$

$$[\![x]\!]^{I}_{g_{[x:=\textcircled{a}]}} \in I(S)$$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$\boldsymbol{I}(a) := \textcircled{a}$	$\boldsymbol{I}(S) := \{\boxed{d}\}$
$\boldsymbol{I}(b) := \textcircled{b}$	$\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$
$\boldsymbol{I}(c) := \textcircled{c}$	$\boldsymbol{g}(x) := \textcircled{b}$
$\boldsymbol{I}(d) := \boxed{d}$	$\boldsymbol{g}(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

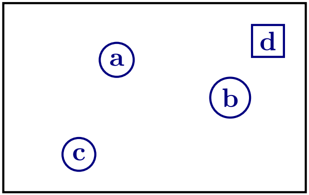
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=\boldsymbol{o}]} \rangle \models Sx$

$$\boldsymbol{g}_{[x:=\textcircled{a}]}(x) \in \boldsymbol{I}(S)$$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(a) := \textcircled{a} \qquad \boldsymbol{I}(S) := \{\boxed{d}\}$$

$$\boldsymbol{I}(b) := \textcircled{b} \qquad \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(c) := \textcircled{c} \qquad \boldsymbol{g}(x) := \textcircled{b}$$

$$\boldsymbol{I}(d) := \boxed{d} \qquad \boldsymbol{g}(y) := \textcircled{a}$$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

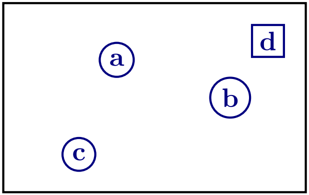
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=o]} \rangle \models Sx$

$$\textcircled{a} \in \boldsymbol{I}(S)$$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$\boldsymbol{I}(a) := \textcircled{a}$	$\boldsymbol{I}(S) := \{\boxed{d}\}$
$\boldsymbol{I}(b) := \textcircled{b}$	$\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$
$\boldsymbol{I}(c) := \textcircled{c}$	$\boldsymbol{g}(x) := \textcircled{b}$
$\boldsymbol{I}(d) := \boxed{d}$	$\boldsymbol{g}(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

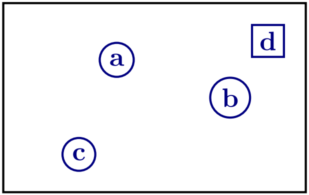
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=o]} \rangle \models Sx$

$$\textcircled{a} \in \{\boxed{d}\}$$

Example

Shapes (***S***quare, ***C***ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(\boldsymbol{a}) := \textcircled{a} \qquad \boldsymbol{I}(\boldsymbol{S}) := \{\boxed{d}\}$$

$$\boldsymbol{I}(\boldsymbol{b}) := \textcircled{b} \qquad \boldsymbol{I}(\boldsymbol{C}) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(\boldsymbol{c}) := \textcircled{c} \qquad \boldsymbol{g}(\boldsymbol{x}) := \textcircled{b}$$

$$\boldsymbol{I}(\boldsymbol{d}) := \boxed{d} \qquad \boldsymbol{g}(\boldsymbol{y}) := \textcircled{a}$$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Ca}$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

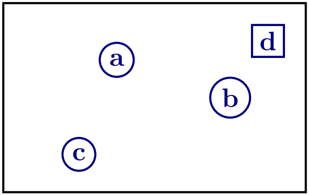
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \boldsymbol{Sx}$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists \boldsymbol{x} \boldsymbol{Sx}$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[\boldsymbol{x}:=\boldsymbol{o}]} \rangle \models \boldsymbol{Sx}$

$\textcircled{a} \in \{\boxed{d}\}$ ✗

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$

$\boldsymbol{I}(a) := \textcircled{a}$ $\quad$ $\boldsymbol{I}(S) := \{\boxed{d}\}$

$\boldsymbol{I}(b) := \textcircled{b}$ $\quad$ $\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$

$\boldsymbol{I}(c) := \textcircled{c}$ $\quad$ $g(x) := \textcircled{b}$

$\boldsymbol{I}(d) := \boxed{d}$ $\quad$ $g(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

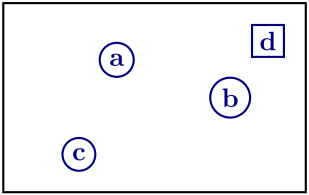
$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, g \rangle \models \exists x Sx$ iff there is a $o \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, g_{[x:=o]} \rangle \models Sx$

...

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$\boldsymbol{I}(a) := \textcircled{a} \qquad \boldsymbol{I}(S) := \{\boxed{d}\}$$

$$\boldsymbol{I}(b) := \textcircled{b} \qquad \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$\boldsymbol{I}(c) := \textcircled{c} \qquad \boldsymbol{g}(x) := \textcircled{b}$$

$$\boldsymbol{I}(d) := \boxed{d} \qquad \boldsymbol{g}(y) := \textcircled{a}$$

$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca \quad \text{iff} \quad \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \ \checkmark$$

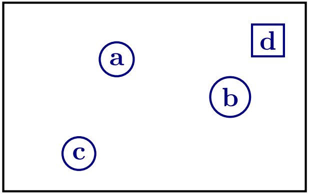
$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx \quad \text{iff} \quad \textcircled{b} \in \{\boxed{d}\} \ \text{✗}$$

$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx \quad \text{iff} \quad \text{there is a } \boldsymbol{o} \in \boldsymbol{D} \text{ such that } \langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=\boldsymbol{o}]} \rangle \models Sx$$

$$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=\boxed{d}]} \rangle \models Sx$$

Example

Shapes (**S**quare, **C**ircle).



$$D := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$I(a) := \textcircled{a} \qquad I(S) := \{\boxed{d}\}$$

$$I(b) := \textcircled{b} \qquad I(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

$$I(c) := \textcircled{c} \qquad g(x) := \textcircled{b}$$

$$I(d) := \boxed{d} \qquad g(y) := \textcircled{a}$$

$\langle D, I, g\rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

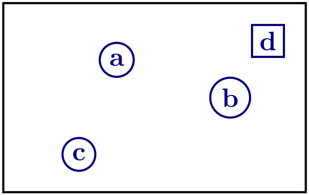
$\langle D, I, g\rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle D, I, g\rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]}\rangle \models Sx$

$$[\![x]\!]^{I}_{g_{[x:=\boxed{d}]}} \in I(S)$$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$\boldsymbol{I}(a) := \textcircled{a}$ $\quad$ $\boldsymbol{I}(S) := \{\boxed{d}\}$

$\boldsymbol{I}(b) := \textcircled{b}$ $\quad$ $\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$

$\boldsymbol{I}(c) := \textcircled{c}$ $\quad$ $g(x) := \textcircled{b}$

$\boldsymbol{I}(d) := \boxed{d}$ $\quad$ $g(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

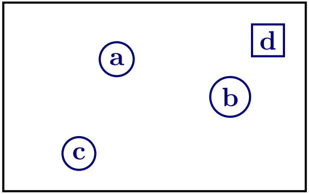
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $o \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=o]} \rangle \models Sx$

$$g_{[x:=\boxed{d}]}(x) \in \boldsymbol{I}(S)$$

Example

Shapes ($\boldsymbol{S}$quare, $\boldsymbol{C}$ircle).



$$\boldsymbol{D} := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$\boldsymbol{I}(a) := \textcircled{a}$	$\boldsymbol{I}(S) := \{\boxed{d}\}$
$\boldsymbol{I}(b) := \textcircled{b}$	$\boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$
$\boldsymbol{I}(c) := \textcircled{c}$	$\boldsymbol{g}(x) := \textcircled{b}$
$\boldsymbol{I}(d) := \boxed{d}$	$\boldsymbol{g}(y) := \textcircled{a}$

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

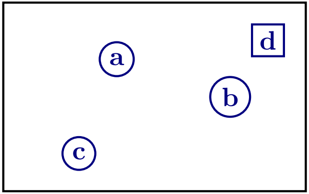
$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g} \rangle \models \exists x Sx$ iff there is a $\boldsymbol{o} \in \boldsymbol{D}$ such that $\langle \boldsymbol{D}, \boldsymbol{I}, \boldsymbol{g}_{[x:=o]} \rangle \models Sx$

$$\boxed{d} \in \boldsymbol{I}(S)$$

Example

Shapes (***S***quare, ***C***ircle).



$$D := \{\textcircled{a}, \textcircled{b}, \textcircled{c}, \boxed{d}\}$$

$$I(a) := \textcircled{a} \qquad I(S) := \{\boxed{d}\}$$

$$I(b) := \textcircled{b} \qquad I(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$$

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$\langle D, I, g \rangle \models Ca$ iff $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\}$ ✓

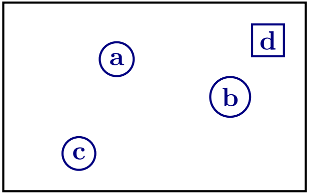
$\langle D, I, g \rangle \models Sx$ iff $\textcircled{b} \in \{\boxed{d}\}$ ✗

$\langle D, I, g \rangle \models \exists x Sx$ iff there is a $o \in D$ such that $\langle D, I, g_{[x:=o]} \rangle \models Sx$

$$\boxed{d} \in \{\boxed{d}\}$$

Example

Shapes (***S***quare, ***C***ircle).



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$$\begin{array}{ll} \boldsymbol{I}(a) := \textcircled{a} & \boldsymbol{I}(S) := \{\boxed{d}\} \\ \boldsymbol{I}(b) := \textcircled{b} & \boldsymbol{I}(C) := \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \\ \boldsymbol{I}(c) := \textcircled{c} & \boldsymbol{g}(x) := \textcircled{b} \\ \boldsymbol{I}(d) := \boxed{d} & \boldsymbol{g}(y) := \textcircled{a} \end{array}$$

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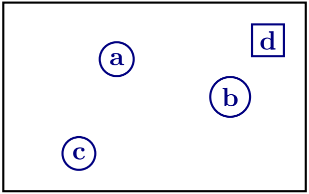

$$\boldsymbol{D} := \{\text{ⓐ}, \text{ⓑ}, \text{ⓒ}, \boxed{\text{d}}\}$$

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- An **inference** $\varphi_1, \ldots, \varphi_n/\psi$ is (logically) **valid** if, for every model M for which we have $M \models \varphi_1$, …, $M \models \varphi_n$, we also have $M \models \psi$. In such case we will write $\varphi_1, \ldots, \varphi_n \models \psi$.

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A formula that can be derived by following these principles in a *finite* number of steps is called a **theorem**.

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Hence, $(\varphi)^x_t \to \exists x \varphi$ is a theorem.

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- $\langle D, I, g\rangle \models t_1 = t_2$ iff $[\![t_1]\!]^I_g$ and $[\![t_2]\!]^I_g$ are the same object.

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- Example: the successor function s is given by $I(s)(n) := n + 1$.