Logic in Action

Chapter 8: Validity Testing

http://www.logicinaction.org/

The *tableau* idea (1)

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

Recall:

An inference is **valid**
iff

The *tableau* idea (1)

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

Recall:

An inference is **valid**
iff
in **every situation** in which **all** premises $\varphi_1, \ldots, \varphi_n$ are true,
ψ is also true.

The *tableau* idea (1)

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

That is:

An inference is **valid**
iff

The *tableau* idea (1)

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

That is:

An inference is **valid**
iff
there is no situation in which **all** premises $\varphi_1, \ldots, \varphi_n$ are true
but ψ is false.

The *tableau* idea (2)

If we can find a situation in which *all* premises $\varphi_1, \ldots, \varphi_n$ are true but ψ is false, then the **inference is not valid**.

The *tableau* idea (2)

If we can find a situation in which *all* premises $\varphi_1, \ldots, \varphi_n$ are true but ψ is false, then the **inference is not valid**.

Let's look for such situations!

We start with something simpler

φ

Recall:

A formula is **valid**
iff

We start with something simpler

φ

Recall:

A formula is **valid**
iff
it is true in **every situation**.

We start with something simpler

φ

That is:

A formula is **valid**
iff

We start with something simpler

φ

That is:

A formula is **valid**
iff
there is no situation in which φ is false.

A very simple case

Is $p \vee q$ valid?

A very simple case

Is $p \vee q$ valid? Can $p \vee q$ be false?

A very simple case

Is $p \vee q$ valid? Can $p \vee q$ be false? If so, how?

A very simple case

Is $p \lor q$ valid? Can $p \lor q$ be false? If so, how?

$\circ \; p \lor q$

A very simple case

Is $p \vee q$ valid? Can $p \vee q$ be false? If so, how?

∘ $p \vee q$

|

∘ p, q

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$\circ$ p, q

Yes!

A very simple case

Is $p \lor q$ valid? Can $p \lor q$ be false? If so, how?

$$\circ \; p \lor q$$

$$|$$

$$\circ \; p, q$$

Yes! Making both p and q **false** makes $p \lor q$ **false**.

A very simple case

Is $p \lor q$ valid? Can $p \lor q$ be false? If so, how?

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$$\circ \; p, q$$

Yes! Making both p and q **false** makes $p \lor q$ **false**.

Hence, $p \lor q$ is not valid.

Another simple case

Is $\neg(p \wedge q)$ valid?

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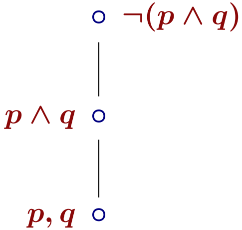

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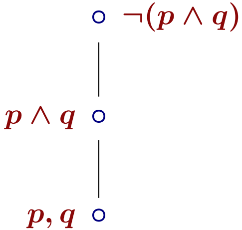

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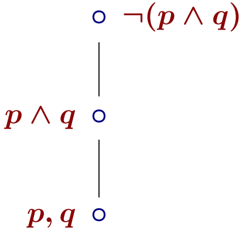

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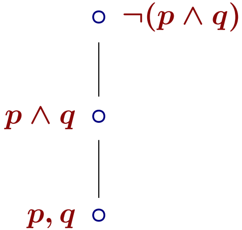

Yes! Making both p and q **true** makes $\neg(p \wedge q)$ **false**.

Hence, $\neg(p \wedge q)$ is not valid.

And another one

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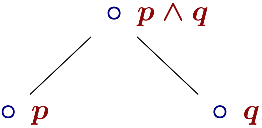

Yes! In fact, there are two ways to make $p \wedge q$ false.

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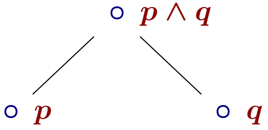

Yes! In fact, there are two ways to make $p \wedge q$ false.

Hence, $p \wedge q$ is not valid.

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Is $\boldsymbol{p} \vee \neg \boldsymbol{p}$ valid? Can $\boldsymbol{p} \vee \neg \boldsymbol{p}$ be false? If so, how?

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$|$

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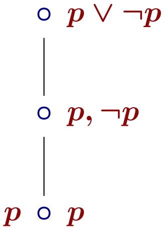

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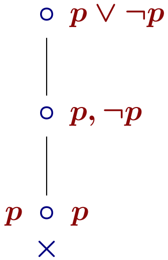

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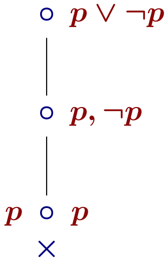

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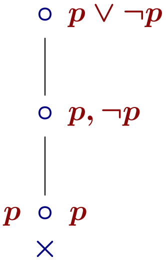

No! We cannot make $p \vee \neg p$ false.

Hence, $p \vee \neg p$ **is valid**.

So, how does this work in general? (1)

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$\neg$		

So, how does this work in general? (1)

$\neg\varphi \circ$

$\neg$

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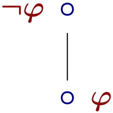
$\neg$



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$\neg$	$\neg\varphi$ ∘ | ∘ φ	∘ $\neg\varphi$

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$\neg$

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$\neg$	$\neg\varphi$ ∘ ∘ φ	∘ $\neg\varphi$ φ ∘
$\wedge$		

So, how does this work in general? (1)

$\neg$	$\neg\varphi \circ$ $\mid$ $\circ\ \varphi$	$\circ\ \neg\varphi$ $\mid$ $\varphi\ \circ$
$\wedge$	$\varphi \wedge \psi \circ$	

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$\neg$	$\neg\varphi \circ$ $\vert$ $\circ\ \varphi$	$\circ\ \neg\varphi$ $\vert$ $\varphi\ \circ$
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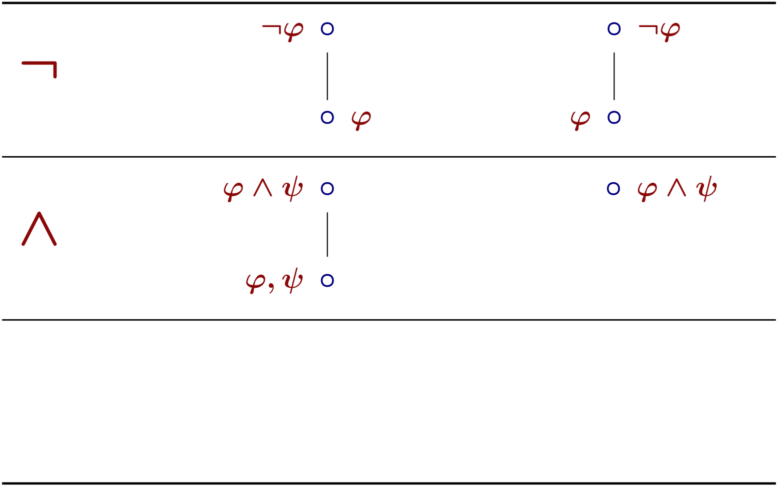

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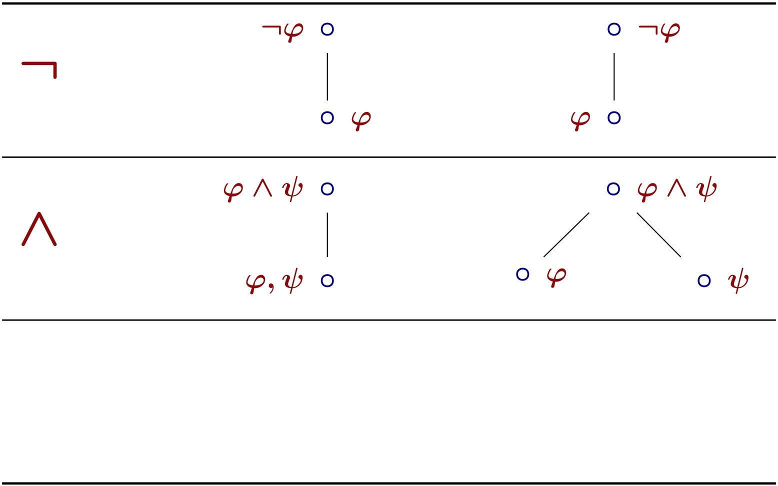

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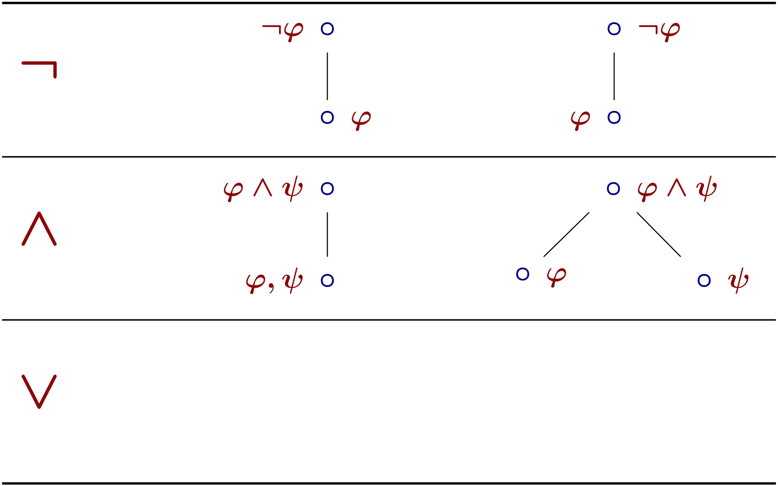

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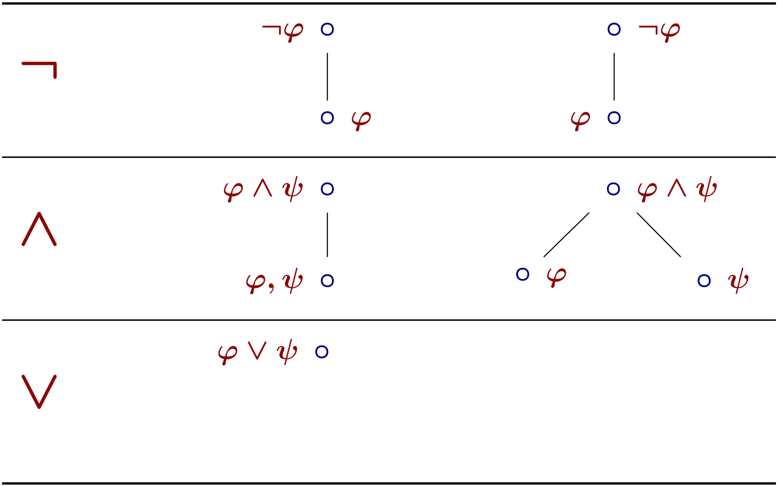

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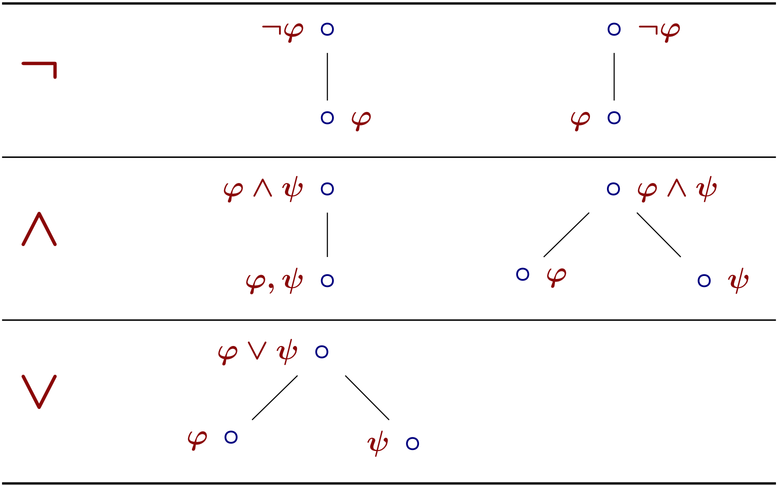

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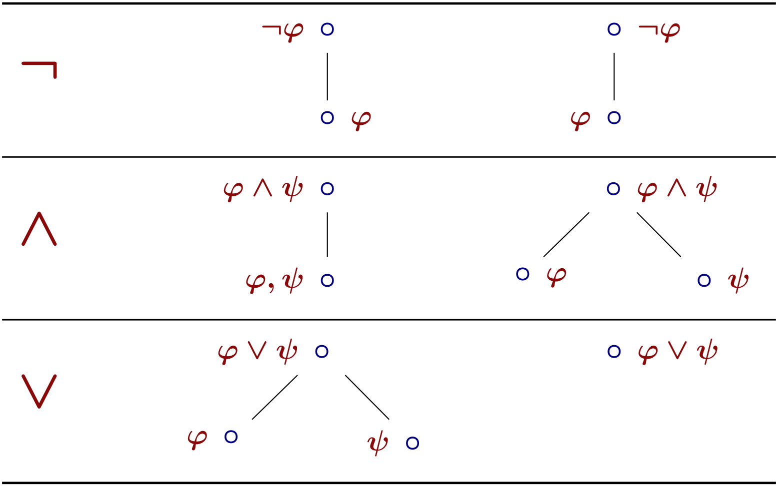

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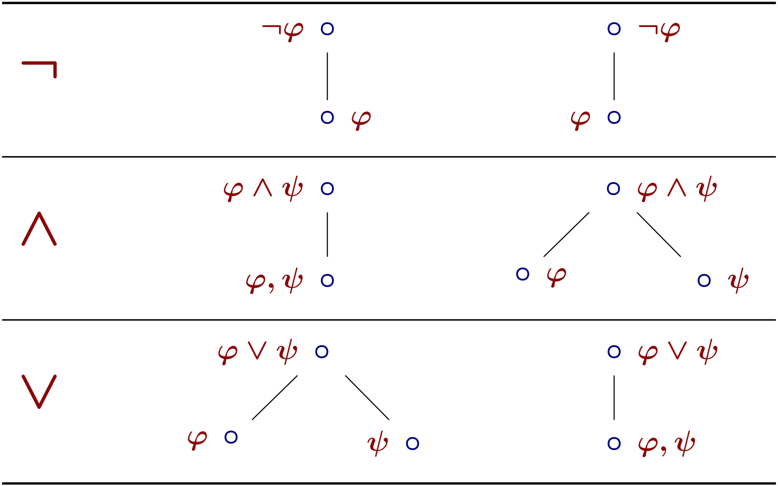

So, how does this work in general? (2)

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$\longrightarrow$

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$\longrightarrow$	$\varphi \to \psi \circ$

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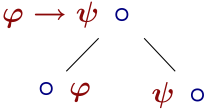
$\longrightarrow$	

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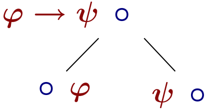
$\longrightarrow$		$\circ \; \varphi \to \psi$

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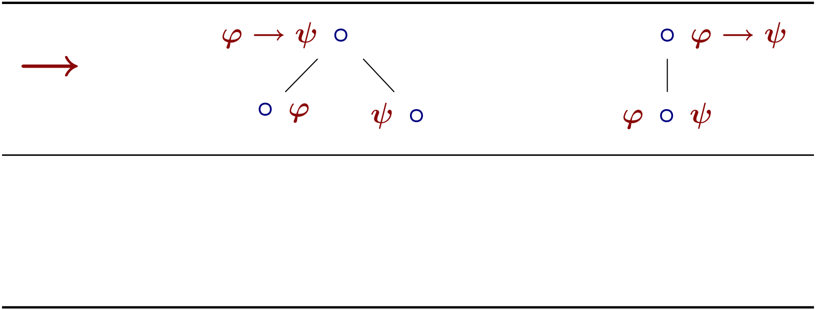

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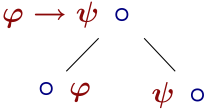
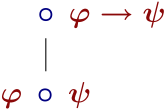
$\longrightarrow$		
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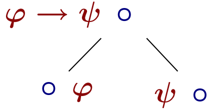
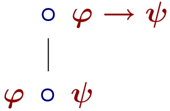
$\longrightarrow$		
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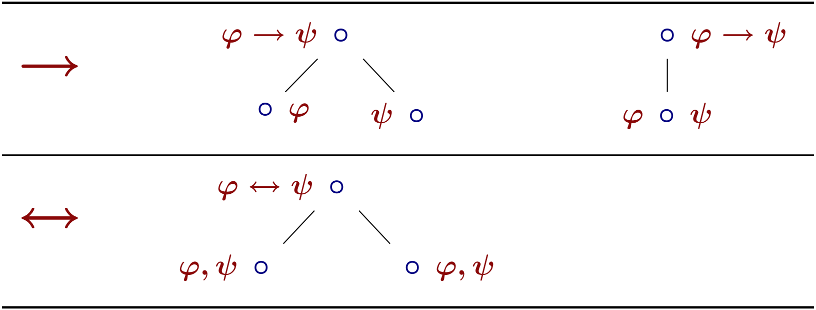

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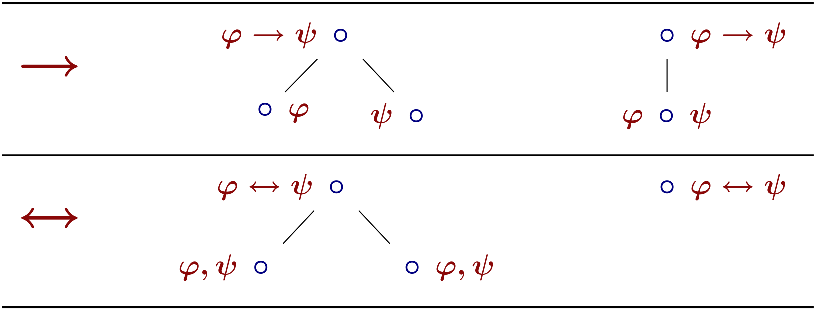

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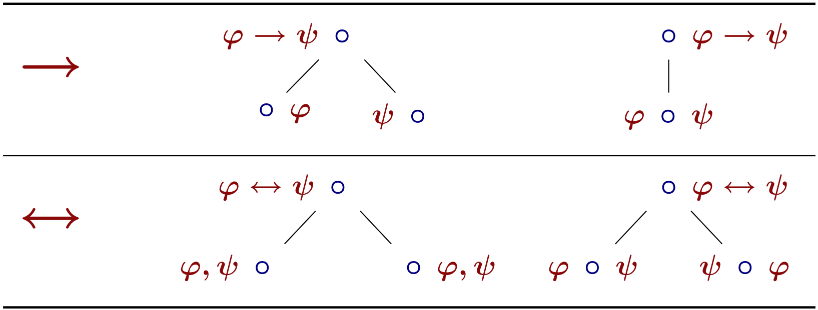

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- **Open tableau**. A *tableau* is **open** if it has *at least* one open branch.

To practice

Decide whether the following formulas are valid or not by using the **tableau** method. In each case, if your answer is *no*, provide a situation in which the formula false (i.e., a *counter-example*).

- $(\neg p) \wedge q$
- $p \rightarrow (q \rightarrow r)$
- $((p \vee q) \rightarrow r) \wedge (p \rightarrow \neg q)$
- $\neg(p \wedge q \wedge r)$
- $(\neg r) \rightarrow (\neg p)$
- $(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow q)$
- $((p \vee q) \vee \neg(p \vee (q \wedge r)))$
- $(p \vee q) \vee \neg(p \vee (q \wedge r))$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $((p \wedge q) \wedge r) \vee ((\neg p \wedge \neg q) \wedge \neg r)$
- $((p \wedge q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$
- $((\neg p \rightarrow q) \wedge (p \vee \neg q)) \rightarrow (p \vee r)$
- $((p \vee q) \rightarrow r) \leftrightarrow ((p \rightarrow r) \vee (q \rightarrow r))$
- $\neg(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$
- $p \vee \neg q$
- $(p \rightarrow q) \rightarrow r$
- $(p \wedge (p \rightarrow q)) \rightarrow q$
- $q \wedge \neg q$
- $\neg\neg p$
- $(q \wedge (p \rightarrow q)) \rightarrow p$
- $((p \leftrightarrow (q \rightarrow r)) \leftrightarrow ((p \leftrightarrow q) \rightarrow r))$
- $\neg((\neg p \vee \neg(q \wedge r)) \vee (p \wedge r))$
- $(p \leftrightarrow (q \rightarrow r)) \leftrightarrow ((p \leftrightarrow q) \rightarrow r)$
- $(p \rightarrow q) \vee (q \rightarrow p)$
- $((\neg p \rightarrow q) \wedge (p \vee \neg q)) \rightarrow p$
- $(p \rightarrow (q \wedge r)) \leftrightarrow ((p \rightarrow q) \wedge (p \rightarrow r))$
- $((p \vee q) \rightarrow r) \leftrightarrow ((p \rightarrow r) \wedge (q \rightarrow r))$
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So we can work with a tableau of the following form

$$\varphi_1, \ldots, \varphi_n \circ \psi$$

To practice

Answer **yes** or **no** to each one of the following questions about the validity of the given inferences. If your answer is **no**, provide a **counter-example**.

- $\varphi \vee \psi, \neg\psi \models \varphi$?
- $\varphi \wedge \neg\varphi \models \psi$?
- $(\varphi \vee \psi) \wedge \chi \models \varphi \vee (\psi \wedge \chi)$?
- $\varphi \rightarrow \psi \models (\neg\varphi) \vee \psi$?
- $\neg\neg\varphi \models \varphi$?
- $\neg(\varphi \wedge \psi), \psi \models \neg\varphi$?
- $((\neg\varphi \vee \neg\psi) \vee \chi), (\psi \vee \chi), \varphi \models \chi$?
- $\neg(\varphi \leftrightarrow \psi) \models \neg\varphi \leftrightarrow \psi$?
- $\varphi \rightarrow (\psi \wedge \chi), \neg((\varphi \vee \psi) \rightarrow \chi) \models \varphi$?
- $\varphi \rightarrow \psi, \varphi \rightarrow \neg\psi \models \neg\varphi$?
- $\varphi \rightarrow \psi \models \psi \rightarrow \varphi$?
- $\varphi \rightarrow \psi, \varphi \models \psi$?
- $\varphi \vee (\psi \wedge \chi) \models (\varphi \vee \psi) \wedge \chi$?
- $\neg(\varphi \wedge \psi) \models \neg\varphi$?
- $\varphi \rightarrow \psi, \psi \models \varphi$?
- $\neg(\psi \wedge \chi), \psi \models \neg\chi$?
- $\varphi \vee \psi, \varphi \rightarrow \chi, \psi \rightarrow \chi \models \chi$?
- $\neg(\varphi \rightarrow (\psi \wedge \chi)), \chi \rightarrow (\varphi \wedge \psi) \models \neg\chi$?
- $\varphi \rightarrow \psi, \varphi \rightarrow \chi \models \psi \leftrightarrow \chi$?
- $\varphi, \psi \models \chi \vee \neg\chi$?
- $\varphi \rightarrow \psi, \chi \rightarrow \eta, \varphi \vee \chi, \neg(\psi \wedge \eta) \models (\psi \rightarrow \varphi) \wedge (\eta \rightarrow \chi)$?

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2. Decide whether a **formula** is **satisfiable** or not (how?), and therefore whether it is a **contradiction** or not.
3. Decide whether a **set of formulas** is **satisfiable** (i.e., all of them can be true) or not (how?).
4. Decide whether an **inference** is **valid**, and therefore whether **two formulas** are **logically equivalent**.

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2. The presented **tableau** method is **complete** for **proving validity** in **propositional** logic: if an inference with propositional formulas is valid, then its tableau will be closed.
3. The presented **tableau** method is **complete** for **finding counterexamples** in **propositional** logic: if an inference with propositional formulas is not valid, then its tableau will have at least one open branch.

Important observations

1. The **tableau** method attempts to build a model (truth-values of atomic propositions) with the specified requirements.
2. The presented **tableau** method is **complete** for **proving validity** in **propositional** logic: if an inference with propositional formulas is valid, then its tableau will be closed.
3. The presented **tableau** method is **complete** for **finding counterexamples** in **propositional** logic: if an inference with propositional formulas is not valid, then its tableau will have at least one open branch.
4. The presented **tableau** method can generate **every counterexample** of an invalid inference in **propositional** logic.

For the predicate logic case

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The tableau method can be used also to decide the validity of inferences in predicate logic.

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We already know how to deal with logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).

We just need to know how to deal with **quantifiers** ($\exists, \forall$).

Quantifiers (1)

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$\exists$

Quantifiers (1)

$\exists x\varphi(x)$ ∘

$\exists$

Quantifiers (1)

$\exists$	$\exists x \varphi(x) \circ$ $\mid$ $\varphi(a) \overset{+}{\circ}$

Quantifiers (1)

$\exists$	$\exists x \varphi(x)$ $\circ$ $\vert$ $\varphi(a)$ $\overset{+}{\circ}$ **For a new** a

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$\exists$	$\exists x \varphi(x)$ ∘ | $\varphi(a)$ $\overset{+}{\circ}$ **For a new** a	∘ $\exists x \varphi(x)$

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$\exists$	$\exists x\varphi(x)\ \circ$ $\vert$ $\varphi(a)\ \overset{+}{\circ}$ **For a new** a	$\circ\ \exists x\varphi(x)$ $\vert$ $\circ\ \varphi(a_1),\ldots,\varphi(a_n)$

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Existential claims:	$\exists x\varphi(x)\ \circ$	$\circ\ \forall x\varphi(x)$
Universal claims:	$\circ\ \exists x\varphi(x)$	$\forall x\varphi(x)\ \circ$

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For a new *a*	**For a new *a***

Quantifiers (3)

Important observation.

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- Every time a new name is introduced ($\overset{+}{\circ}$), we should **reactivate** every previous universal claim.

Recommendations

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When working with predicate tableau, try to follow this order:

1. Work with logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).
2. Then, when working with existential claims.
3. Finally work with universal claims.

To practice

Which of the following statements are true?

- $\forall x(Px) \models \neg\exists x(\neg Px)$
- $\neg\exists x(Px) \models \forall x(\neg Px)$
- $\forall x\exists yRxy \models \forall xRxx$
- $\forall x\forall yRxy \models \forall xRxx$
- $\forall x\forall yRxy, Rab \models Raa$
- $\forall x(Px \rightarrow Qx) \vee \forall y(Qy \rightarrow Py) \models \forall x\forall y((Px \wedge Qy) \rightarrow (Qx \vee Py))$
- $\forall xPx \rightarrow \forall xQx \models \forall x(Px \rightarrow Qx)$
- $\forall x(Px \rightarrow Qx) \models \forall xPx \rightarrow \forall xQx$
- $\exists y\forall xRxy \models \forall x\exists yRxy$
- $\forall x(Px \rightarrow Qx), \exists x(Px \wedge Rx) \models \exists x(Qx \wedge Rx)$
- $\forall x(Px \rightarrow Qx), \exists x(\neg Px \wedge Rx) \models \exists x(\neg Qx \wedge Rx)$
- $\neg\exists x(Px \wedge Qx), \forall x(Qx \rightarrow Rx) \models \neg\exists x(Px \wedge Rx)$
- $\forall x(Px \rightarrow Qx), \forall x(Qx \rightarrow Rx), \forall x(Rx \rightarrow Px) \models \forall x(Qx \wedge Px)$

Can we always find a counterexample?

Consider the following inference

$$\frac{\forall y \exists x \boldsymbol{Rxy}}{\exists y \forall x \boldsymbol{Rxy}}$$

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- What does the inference says?

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$$\frac{\forall y \exists x Rxy}{\exists y \forall x Rxy}$$

- What does the inference says?
- Is it valid?

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What can we do?

The problem

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- But maybe one of the previous names is useful.

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The solution

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- For existential claims, we always introduce a new name.
- But maybe one of the previous names is useful.

The solution

- For existential claims, we will now consider the possibility of a previous name being the adequate one.

Extended rules for existential claims

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$\exists x \varphi(x)$ ∘

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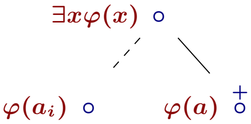

Extended rules for existential claims

$\exists x \varphi(x)$ ∘

$\varphi(a_i)$ ∘ $\qquad$ $\varphi(a)$ $\overset{+}{\circ}$

For an existing a_i and a new a

Extended rules for existential claims

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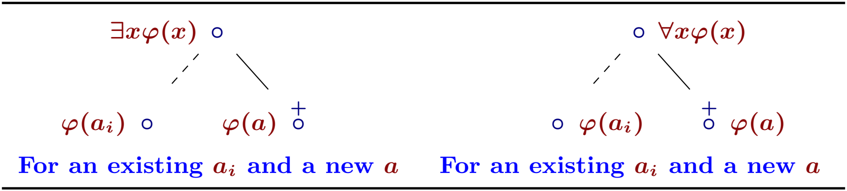

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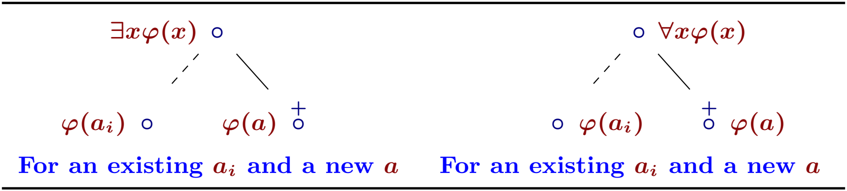

What happen now with $\dfrac{\forall y \exists x Rxy}{\exists y \forall x Rxy}$?

Can we always find a counterexample?

Consider the following inference

$$\frac{\forall y \exists x Rxy, \forall x \forall y \forall z \big((Rxy \wedge Ryz) \rightarrow Rxz\big)}{\exists x \exists y (Rxy \wedge Ryx)}$$

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- Hence, every model we built is **finite**.
- There are invalid inferences whose counterexamples are **infinite** models.

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We will introduce tableau rules for the case with a **single equivalence** (i.e., *reflexive*, *transitive* and *symmetric*) **relation** R.

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- given our assumptions (a *unique equivalence* relation), our domain is just a set of worlds (i.e., every world is accessible from every other).
- Hence, each one of our nodes will have the information for this set of worlds.

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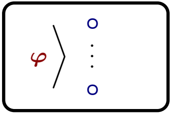

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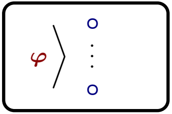

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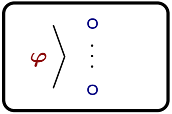

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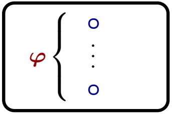

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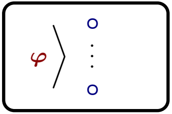

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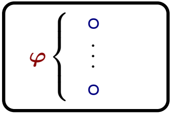

- Analogous for the right side.

How rules for connectives work now (1)

For negation $\neg$:

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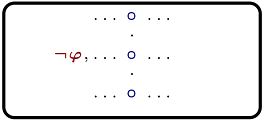

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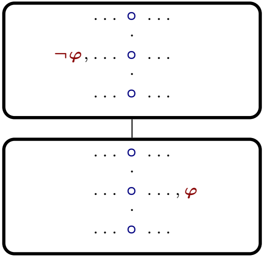

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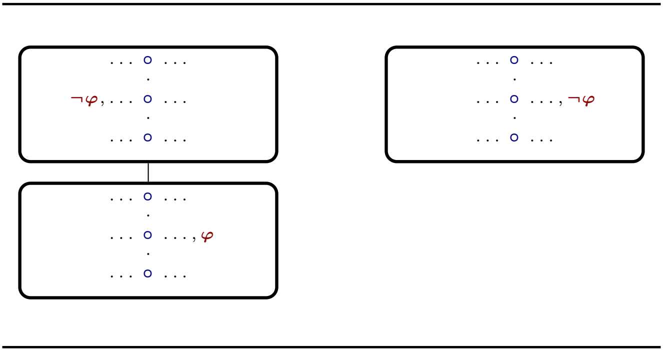

How rules for connectives work now (1)

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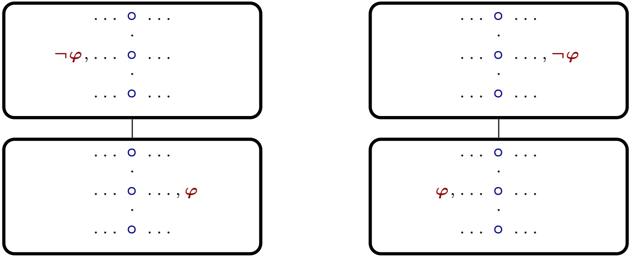

How rules for connectives work now (2)

For conjunction $\wedge$:

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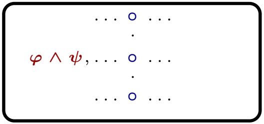

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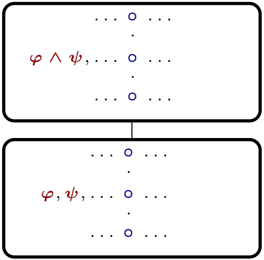

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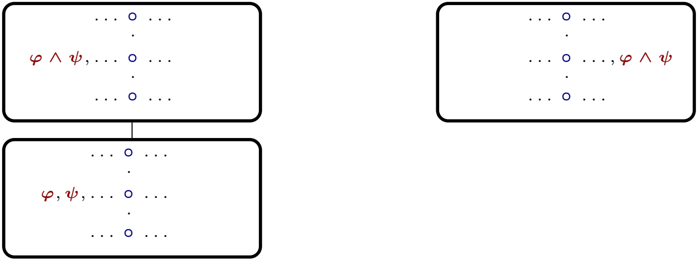

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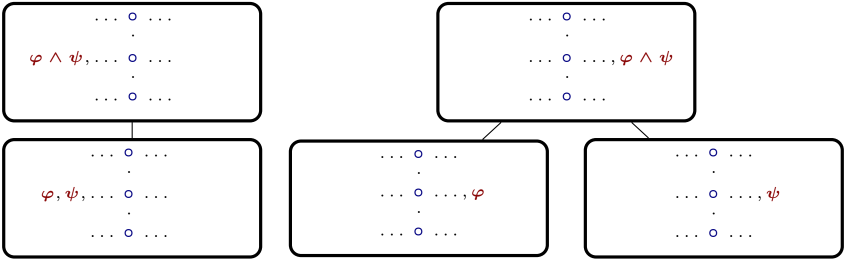

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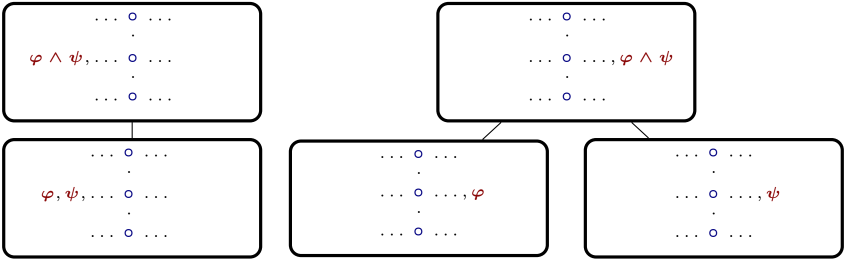

And so on . . .

For the modality (1)

For the modality $\Box$:

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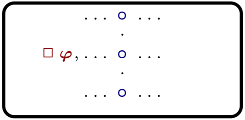

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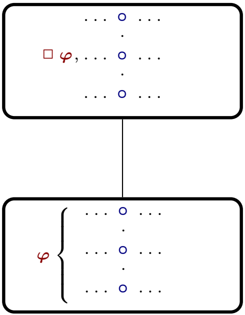

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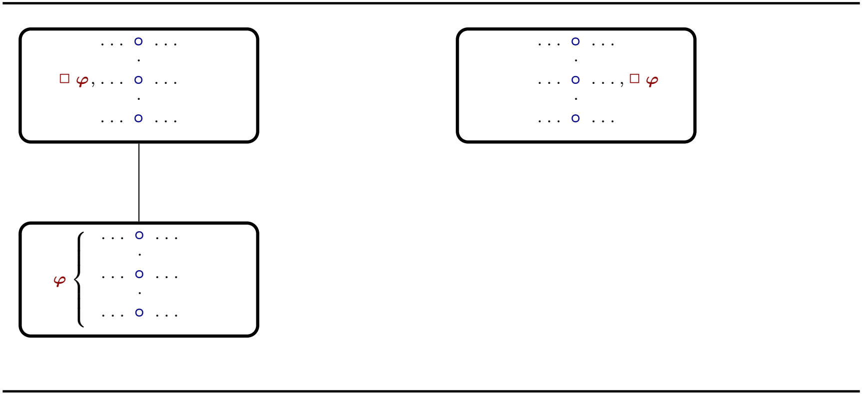

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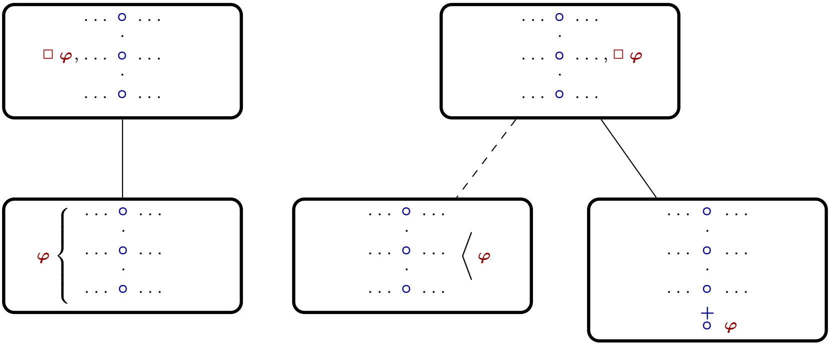

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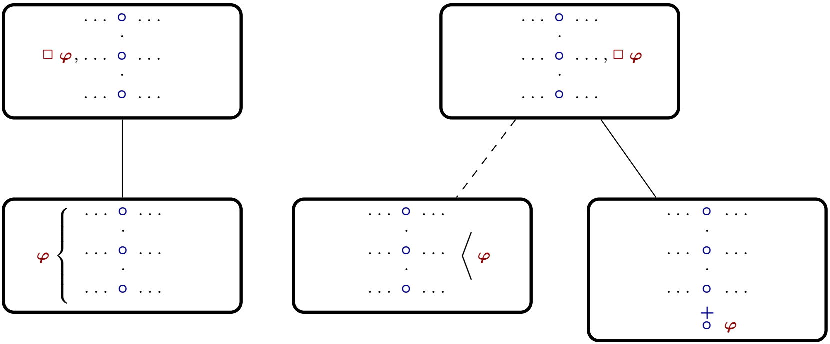

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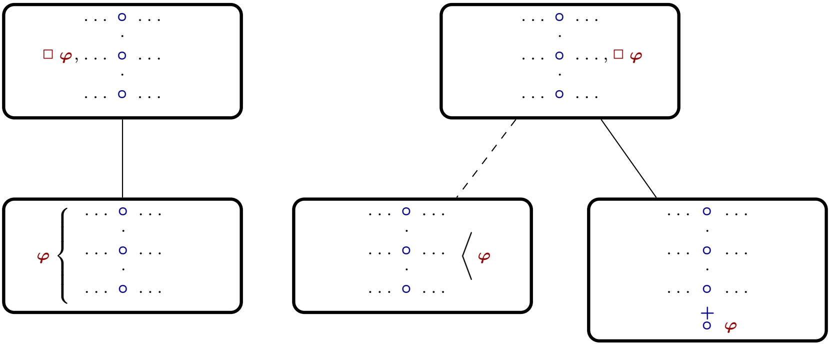

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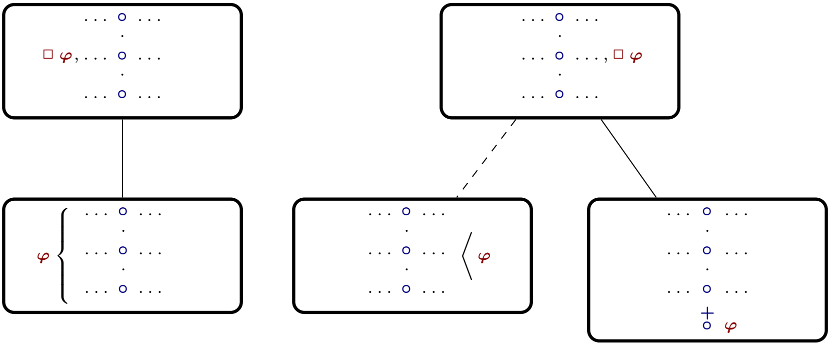

The intuition:

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- if $\Box\varphi$ is *false*, then at least one world should make φ false.

For the modality (2)

Some terminology:

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Universal claim:

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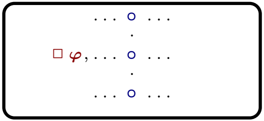

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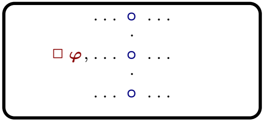

Existential claim:

For the modality (2)

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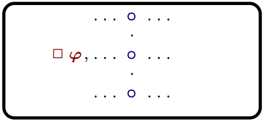
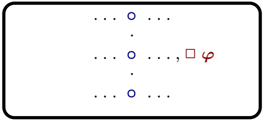
Universal claim:	
Existential claim:	

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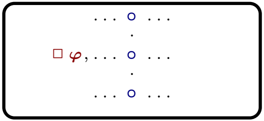
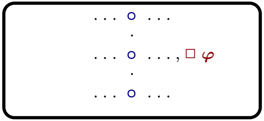
Universal claim:	
Existential claim:	

Important observation.

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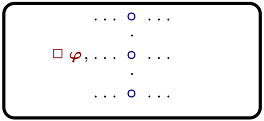
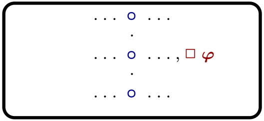
Universal claim:	
Existential claim:	

Important observation.

- Every time a new world is introduced ($\overset{+}{\circ}$), we should **reactivate** every previous universal claim.

Closed/open branch/tableau

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- **Closed branch**. A *branch* is **closed** if in its end multi-sequent *there is a sequent* in which *there is a formula* that appears on both the left and the right side.

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- **Open tableau**. A *tableau* is **open** if it has *at least* one open branch.

To practice

Answer **yes** or **no** to each one of the following questions about the validity of the given inferences. If your answer is **no**, provide a **counter-example**.

- $\Box(\varphi \wedge \psi) \models \Box\varphi \wedge \Box\psi$?
- $\Box\varphi \wedge \Box\psi \models \Box(\varphi \wedge \psi)$?
- $\Box(\varphi \vee \psi) \models \Box\varphi \vee \Box\psi$?
- $\Box\varphi \vee \Box\psi \models \Box(\varphi \vee \psi)$?
- $\Box\varphi \models \Box\Box\varphi$?
- $\Box\varphi \models \neg\Box\neg\varphi$?
- $\varphi \models \Box\neg\Box\neg\varphi$?
- $\Box\varphi \models \varphi$?

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4. The presented **tableau** method can generate **every counterexample** of an invalid inference in **epistemic** logic *with a single equivalence relation*.