Logic in Action

Chapter 9: Proofs

http://www.logicinaction.org/

Systems revised so far

Issues with the **tableau** method.

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- It is a *refutation* method.

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Issues with the **tableau** method.

- It is a *refutation* method.
- It does not follow the way humans reason.

Issues with the **presented derivation systems**.

- Proofs are not very natural (e.g., try to prove $\varphi \rightarrow \neg\neg\varphi$).
- They do not facilitate *conditional* reasoning.

The *deduction* property

The *deduction* property

$$\Sigma, \varphi \models \psi \qquad \text{if and only if} \qquad \Sigma \models \varphi \rightarrow \psi$$

What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.

What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

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- But if we can make assumptions …

What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions ...

$$
\begin{array}{ll}
1 & \left|\ \underline{\varphi}\right.
\end{array}
$$

What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions ...

1	φ	
2	φ	repetition 1

What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions …

1	$\mid \varphi$	
2	$\mid \varphi$	`repetition 1`
3	$\varphi \rightarrow \varphi$	`deduction 1-2`

What if we can make assumptions?

Consider a proof for $\varphi \longrightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions ...

1	φ (assumption)	
2	φ	`repetition 1`
3	$\varphi \longrightarrow \varphi$	`deduction 1-2`

This is the main idea for **the deduction rule**.

The deduction rule

Suppose you want to prove $\varphi \to \psi$.

The deduction rule

Suppose you want to prove $\varphi \to \psi$.

- Assume φ.

φ

The deduction rule

Suppose you want to prove $\varphi \to \psi$.

- Assume φ.
- If after further steps

$$\begin{array}{|l} \varphi \\ \hline \vdots \end{array}$$

The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume φ.
- If after further steps
- you can prove ψ,

$$\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}$$

The **deduction** rule

Suppose you want to prove $\varphi \to \psi$.

- Assume φ.
- If after further steps
- you can prove ψ,
- then you actually have $\varphi \to \psi$.

$$\dfrac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi}\ \texttt{deduction}$$

Recall

The three axioms for propositional logic

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The three axioms for propositional logic

1. $\varphi \to (\psi \to \varphi)$

Recall

The three axioms for propositional logic

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
2. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

Recall

The three axioms for propositional logic

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

Proving the axioms (1)

The axiom

$$\varphi \to (\psi \to \varphi)$$

can be proved from *deduction*:

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$$\begin{array}{ll} 1 & \underline{|\ \varphi} \end{array}$$

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

$$\begin{array}{ll} 1 & \mid \ \varphi \\ 2 & \mid \ \mid \ \psi \end{array}$$

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1	φ	
2	$\quad \psi$	
3	$\quad \varphi$	repetition 1

Proving the axioms (1)

The axiom

$$\varphi \to (\psi \to \varphi)$$

can be proved from *deduction*:

1	φ	
2	$\quad \psi$	
3	$\quad \varphi$	`repetition 1`
4	$\psi \to \varphi$	`deduction 2-3`

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1	φ	
2	ψ	
3	φ	repetition 1
4	$\psi \rightarrow \varphi$	deduction 2-3
5	$\varphi \rightarrow (\psi \rightarrow \varphi)$	deduction 1-4

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

$$\begin{array}{ll} 1 & \underline{\mid\ \varphi \to (\psi \to \chi)} \end{array}$$

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$
2	$\quad \varphi \to \psi$

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

$$\begin{array}{lll}
1 & & \varphi \to (\psi \to \chi) \\
2 & & \quad \varphi \to \psi \\
3 & & \quad\quad \varphi
\end{array}$$

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\quad \varphi \to \psi$	
3	$\quad\quad \varphi$	
4	$\quad\quad \psi$	modus ponens 3,2

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\quad \varphi \to \psi$	
3	$\quad\quad \varphi$	
4	$\quad\quad \psi$	`modus ponens 3,2`
5	$\quad\quad \psi \to \chi$	`modus ponens 3,1`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \to \chi$	`modus ponens 3,1`
6	χ	`modus ponens 4,5`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\quad \varphi \to \psi$	
3	$\quad\quad \varphi$	
4	$\quad\quad \psi$	`modus ponens 3,2`
5	$\quad\quad \psi \to \chi$	`modus ponens 3,1`
6	$\quad\quad \chi$	`modus ponens 4,5`
7	$\quad \varphi \to \chi$	`deduction 3-6`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\quad \varphi \to \psi$	
3	$\quad\quad \varphi$	
4	$\quad\quad \psi$	modus ponens 3,2
5	$\quad\quad \psi \to \chi$	modus ponens 3,1
6	$\quad\quad \chi$	modus ponens 4,5
7	$\quad \varphi \to \chi$	deduction 3-6
8	$(\varphi \to \psi) \to (\varphi \to \chi)$	deduction 2-7

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\quad \varphi \to \psi$	
3	$\quad\quad \varphi$	
4	$\quad\quad \psi$	modus ponens 3,2
5	$\quad\quad \psi \to \chi$	modus ponens 3,1
6	$\quad\quad \chi$	modus ponens 4,5
7	$\quad \varphi \to \chi$	deduction 3-6
8	$(\varphi \to \psi) \to (\varphi \to \chi)$	deduction 2-7
9	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	deduction 1-8

We need more

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

cannot be proved from *modus ponens* and *deduction*.

We need more

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

cannot be proved from *modus ponens* and *deduction*.

We need a way to deal with negations.

The refutation rule

Suppose you want to prove φ.

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.

$\neg\varphi$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps

$$\neg\varphi$$

$$\vdots$$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\bot$,

$$
\begin{array}{|l}
\neg\varphi \\
\hline
\vdots \\
\bot
\end{array}
$$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\perp$,
- then $\neg\varphi$ cannot be true

$$\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \perp \end{array}$$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\bot$,
- then $\neg\varphi$ cannot be true
- so you actually have φ.

$$\cfrac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\ \texttt{refutation}$$

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

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The axiom

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can be proved from *modus ponens*, *deduction* and *refutation*:

$$\begin{array}{ll} 1 & \neg\varphi \to \neg\psi \end{array}$$

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

$$\begin{array}{ll} 1 & \quad \neg\varphi \to \neg\psi \\ 2 & \qquad \psi \end{array}$$

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$
2	$\quad \psi$
3	$\quad\quad \neg\varphi$

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$	
2	$\quad\psi$	
3	$\quad\quad\neg\varphi$	
4	$\quad\quad\neg\psi$	`modus ponens 3,1`
5	$\quad\quad\bot$	`modus ponens 2,4`

Proving the axioms (3)

The axiom

$$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \rightarrow \neg\psi$	
2	$\quad \psi$	
3	$\quad\quad \neg\varphi$	
4	$\quad\quad \neg\psi$	`modus ponens 3,1`
5	$\quad\quad \bot$	`modus ponens 2,4`
6	$\quad \varphi$	`refutation 3-5`

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$	
2	$\quad \psi$	
3	$\quad\quad \neg\varphi$	
4	$\quad\quad \neg\psi$	modus ponens 3,1
5	$\quad\quad \bot$	modus ponens 2,4
6	$\quad \varphi$	refutation 3-5
7	$\psi \to \varphi$	deduction 2-6

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`
5	$\bot$	`modus ponens 2,4`
6	φ	`refutation 3-5`
7	$\psi \to \varphi$	`deduction 2-6`
8	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	`deduction 1-7`

Proving the axioms (3)

The axiom

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can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\neg\varphi \to \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	modus ponens 3,1
5	$\bot$	modus ponens 2,4
6	φ	refutation 3-5
7	$\psi \to \varphi$	deduction 2-6
8	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	deduction 1-7

For step 5, note that $\neg\psi$ can be seen as an abbreviation of $\psi \to \bot$.

So . . .

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}\quad \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi}\quad \texttt{deduction} \qquad\qquad \frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\quad \texttt{refutation}$$

So ...

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}\ \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi}\ \texttt{deduction} \qquad \frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\ \texttt{refutation}$$

The *modus ponens*, *deduction* and *refutation* rules are a complete system for propositional logic.

To facilitate things . . .

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- Natural deduction introduces rules to manipulate all the connectives in an easy way.

For implication $\to$

For implication $\to$

$$\frac{\varphi \,,\, \varphi \to \psi}{\psi} \quad \texttt{modus ponens}$$

For implication $\rightarrow$

$$\frac{\varphi \, , \, \varphi \rightarrow \psi}{\psi} \quad \texttt{modus ponens}$$

$$\mathbf{E}_{\rightarrow}$$

For implication $\to$

$$\frac{\varphi \,,\, \varphi \to \psi}{\psi} \quad \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi} \quad \texttt{deduction}$$

$\mathbf{E}_{\to}$

For implication $\longrightarrow$

$$\frac{\varphi\,,\ \varphi \longrightarrow \psi}{\psi} \quad \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \longrightarrow \psi} \quad \texttt{deduction}$$

$\mathbf{E}_{\longrightarrow}$ $\qquad$ $\mathbf{I}_{\longrightarrow}$

For negation $\neg$

For negation $\neg$

$$\frac{\neg\varphi\ ,\ \varphi}{\bot}$$

For negation $\neg$

$$\frac{\neg\varphi \, , \, \varphi}{\bot}$$

$\mathbf{E}_{\neg}$

For negation $\neg$

$$\frac{\neg\varphi\ ,\ \varphi}{\bot}$$

$$\frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\ \texttt{refutation}$$

$\mathbf{E}_{\neg}$

For negation $\neg$

$$\frac{\neg\varphi\ ,\ \varphi}{\bot}$$

$$\frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\ \texttt{refutation}$$

$\mathbf{E}_{\neg}$ $\mathbf{I}_{\neg}$

For conjunction $\wedge$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

$$\mathbf{E}_{\wedge}$$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

$$\frac{\varphi\ ,\ \psi}{\varphi \wedge \psi}$$

$\mathbf{E}_{\wedge}$

For conjunction $\wedge$

$\mathbf{E}_\wedge$	$\mathbf{I}_\wedge$
$$\frac{\varphi \wedge \psi}{\varphi}$$ $$\frac{\varphi \wedge \psi}{\psi}$$	$$\frac{\varphi \, , \, \psi}{\varphi \wedge \psi}$$

For disjunction $\vee$

For disjunction $\vee$

$$\cfrac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

For disjunction $\vee$

$$
\cfrac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}
$$

$\mathbf{E}_{\vee}$

For disjunction $\vee$

$$\dfrac{\varphi \vee \psi\,, \quad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \quad , \quad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

$$\dfrac{\varphi}{\varphi \vee \psi} \qquad \dfrac{\psi}{\varphi \vee \psi}$$

$\mathbf{E}_{\vee}$

For disjunction $\vee$

$$\dfrac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

$$\dfrac{\varphi}{\varphi \vee \psi} \qquad \dfrac{\psi}{\varphi \vee \psi}$$

$\mathbf{E}_{\vee}$	$\mathbf{I}_{\vee}$

For *predicate* logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

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- **Bounded variable**.

For *predicate* logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

- **Bounded variable**.
- **Substitution of a variable for a term in a formula**.

Bounded variable

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- **Scope of a quantifier.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the subformula φ is said to be **the scope** of the quantifier $\forall$ ($\exists$).

Bounded variable

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- **Binding a variable.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the quantifier $\forall$ ($\exists$) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ.

Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x \varphi$ ($\exists x \varphi$), the subformula φ is said to be **the scope** of the quantifier $\forall$ ($\exists$).

- **Binding a variable.** In a formula of the form $\forall x \varphi$ ($\exists x \varphi$), the quantifier $\forall$ ($\exists$) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ.

- **Bounded variable**. An occurrence of a variable x is **bounded** in a formula φ if there is a quantifier in φ that binds it.

Substitution (1)

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- **Substitution inside a <u>term</u>.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)^y_t$$

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- **Substitution inside a <u>term</u>.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)^y_t$$

- Formally,

$$\text{For a } \textbf{constant}\text{:} \qquad (c)^y_t := c$$

$$\text{For a } \textbf{variable}\text{:} \qquad \begin{cases} (x)^y_t := x & \text{for } x \text{ different from } y \\ (y)^y_t := t \end{cases}$$

Substitution (1)

- **Substitution inside a <u>term</u>.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)^y_t$$

- Formally,

For a **constant**: $(c)^y_t := c$

For a **variable**: $\begin{cases} (x)^y_t := x & \text{for } x \text{ different from } y \\ (y)^y_t := t & \end{cases}$

Examples:

$$(a)^x_c := a$$
$$(x)^y_a := x$$
$$(z)^z_y := y$$

Substitution (2)

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- **Substitution inside a <u>formula</u>.** Replacing the **free** occurrences of the variable y for the term t inside the **formula** φ produces the **formula** denoted by

$$(\varphi)^y_t$$

Substitution (2)

- **Substitution inside a formula.** Replacing the **free** occurrences of the variable y for the term t inside the **formula** φ produces the **formula** denoted by

$$(\varphi)^y_t$$

- Formally,

$$\begin{aligned}
(Pt_1 \cdots t_n)^y_t &:= P(t_1)^y_t \cdots (t_n)^y_t \\
(\neg\varphi)^y_t &:= \neg(\varphi)^y_t \\
(\varphi \wedge \psi)^y_t &:= (\varphi)^y_t \wedge (\psi)^y_t \\
(\varphi \vee \psi)^y_t &:= (\varphi)^y_t \vee (\psi)^y_t \\
(\varphi \to \psi)^y_t &:= (\varphi)^y_t \to (\psi)^y_t \\
(\varphi \leftrightarrow \psi)^y_t &:= (\varphi)^y_t \leftrightarrow (\psi)^y_t
\end{aligned}$$

$$\begin{cases} (\forall x\varphi)^y_t := \forall x(\varphi)^y_t \\ (\forall y\varphi)^y_t := \forall y\varphi \end{cases}$$

$$\begin{cases} (\exists x\varphi)^y_t := \exists x(\varphi)^y_t \\ (\exists y\varphi)^y_t := \exists y\varphi \end{cases}$$

For the universal quantifier ∀

For the universal quantifier $\forall$

$$\frac{\forall x\, \varphi}{(\varphi)^x_t}$$

provided that no variable in t occurs bounded in φ

For the universal quantifier $\forall$

$$\frac{\forall x\, \varphi}{(\varphi)^x_t}$$

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$\mathbf{E}_\forall$

For the universal quantifier $\forall$

$$\frac{\forall x\, \varphi}{(\varphi)^x_t}$$

provided that no variable in t occurs bounded in φ

$\mathbf{E}_\forall$

$$\begin{array}{l} u \;\left|\; \begin{array}{l} \hline \quad\vdots \\ (\varphi)^x_u \end{array}\right. \\ \hline \qquad \forall x\, \varphi \end{array}$$

for u a special symbol not used anywhere else in the proof

For the universal quantifier $\forall$

$$\frac{\forall x\,\varphi}{(\varphi)^x_t}$$

provided that no variable in t occurs bounded in φ

$\mathbf{E}_\forall$

$$\frac{\begin{array}{r|l} u & \\ \hline & \vdots \\ & (\varphi)^x_u \end{array}}{\forall x\,\varphi}$$

for u a special symbol not used anywhere else in the proof

$\mathbf{I}_\forall$

For the existential quantifier ∃

For the existential quantifier $\exists$

$$
\dfrac{\exists x\,\varphi\;, \qquad \begin{array}{r|l} u & (\varphi)^x_u \\ \hline & \vdots \\ & \psi \end{array}}{\psi}
$$

for u a special symbol not used anywhere in the proof

For the existential quantifier $\exists$

$$\dfrac{\exists x\,\varphi\ , \qquad \begin{array}{l|l} u & (\varphi)^x_u \\ & \quad\vdots \\ & \quad\psi \end{array}}{\psi}$$

for u a special symbol not used anywhere in the proof

$\mathbf{E}_\exists$

For the existential quantifier $\exists$

$$\frac{\exists x\,\varphi\,, \qquad \begin{array}{r|l} u & (\varphi)^x_u \\ \cline{2-2} & \vdots \\ & \psi \end{array}}{\psi}$$

for u a special symbol not used anywhere in the proof

$\mathbf{E}_{\exists}$

$$\frac{(\varphi)^x_t}{\exists x\,\varphi}$$

provided that no variable in t occurs bounded in φ

For the existential quantifier $\exists$

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$\mathbf{I}_\exists$

For the identity symbol =

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$$\frac{t_1 = t_2 \; , \; \varphi}{\varphi_{[t_1/t_2]}}$$

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$\mathbf{E}_=$

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for any term t.

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