

Logic in Action

Chapter 9: Proofs

`http://www.logicinaction.org/`

Systems revised so far

Issues with the **tableau** method.

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- It is a *refutation* method.

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- Proofs are not very natural (e.g., try to prove $\varphi \rightarrow \neg\neg\varphi$).

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Issues with the **tableau** method.

- It is a *refutation* method.
- It does not follow the way humans reason.

Issues with the **presented derivation systems**.

- Proofs are not very natural (e.g., try to prove $\varphi \rightarrow \neg\neg\varphi$).
- They do not facilitate *conditional* reasoning.

The *deduction* property

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$\Sigma, \varphi \models \psi$ if and only if $\Sigma \models \varphi \rightarrow \psi$

What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

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- But if we can make assumptions ...

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- But if we can make assumptions ...

1 $\left[\begin{array}{l} \varphi \end{array} \right.$

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- But if we can make assumptions ...

1	φ	
	┌	
2	φ	repetition 1

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1	φ	
2	φ	repetition 1
3	$\varphi \rightarrow \varphi$	deduction 1-2

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- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions ...

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	├──	
2	φ	repetition 1
3	$\varphi \rightarrow \varphi$	deduction 1-2

This is the main idea for **the deduction rule**.

The deduction rule

Suppose you want to prove $\varphi \rightarrow \psi$.

The deduction rule

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- Assume φ .

φ

The deduction rule

Suppose you want to prove $\varphi \rightarrow \psi$.

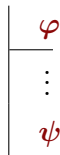
- Assume φ .
- If after further steps



The deduction rule

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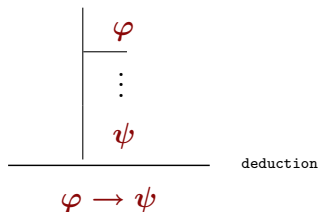
- Assume φ .
- If after further steps
- you can prove ψ ,



The deduction rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume φ .
- If after further steps
- you can prove ψ ,
- then you actually have $\varphi \rightarrow \psi$.



Recall

The three axioms for propositional logic

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1 $\varphi \rightarrow (\psi \rightarrow \varphi)$

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The three axioms for propositional logic

$$\textcircled{1} \quad \varphi \rightarrow (\psi \rightarrow \varphi)$$

$$\textcircled{2} \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

Recall

The three axioms for propositional logic

$$\textcircled{1} \quad \varphi \rightarrow (\psi \rightarrow \varphi)$$

$$\textcircled{2} \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

$$\textcircled{3} \quad (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

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$$\begin{array}{l} 1 \\ 2 \end{array} \left| \begin{array}{l} \varphi \\ \psi \end{array} \right.$$

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1	<div style="border-bottom: 1px solid black; display: inline-block; width: 1em; margin-right: 0.5em;"></div> φ
2	<div style="border-left: 1px solid black; padding-left: 5px; vertical-align: middle;"> <div style="border-bottom: 1px solid black; display: inline-block; width: 1em; margin-right: 0.5em;"></div> ψ </div>
3	<div style="border-left: 1px solid black; padding-left: 5px; vertical-align: middle;"> φ </div>

repetition 1

Proving the axioms (1)

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1	φ	
	└───	
2	ψ	
	└───	
3	φ	repetition 1
4	$\psi \rightarrow \varphi$	deduction 2-3

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The axiom

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1	φ	
2	ψ	
3	φ	repetition 1
4	$\psi \rightarrow \varphi$	deduction 2-3
5	$\varphi \rightarrow (\psi \rightarrow \varphi)$	deduction 1-4

Proving the axioms (2)

The axiom

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

can be proved from *modus ponens* and *deduction*:

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 \end{array}$$

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2	$\varphi \rightarrow \psi$	
3	φ	
4	ψ	modus ponens 3,2

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4	ψ	<i>modus ponens</i> 3,2
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5	$\psi \rightarrow \chi$	modus ponens 3,1
6	χ	modus ponens 4,5

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7	$\varphi \rightarrow \chi$	deduction 3-6

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6	χ	modus ponens 4,5
7	$\varphi \rightarrow \chi$	deduction 3-6
8	$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$	deduction 2-7

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6	χ	modus ponens 4,5
7	$\varphi \rightarrow \chi$	deduction 3-6
8	$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$	deduction 2-7
9	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	deduction 1-8

We need more

The axiom

$$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

cannot be proved from *modus ponens* and *deduction*.

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cannot be proved from *modus ponens* and *deduction*.

We need a way to deal with negations.

The refutation rule

Suppose you want to prove φ .

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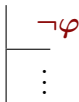
- Assume $\neg\varphi$.

\perp

The refutation rule

Suppose you want to prove φ .

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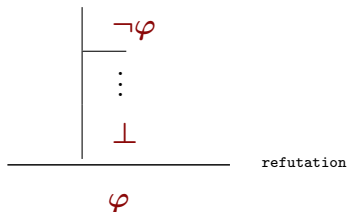
- Assume $\neg\varphi$.
- If after further steps
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- then $\neg\varphi$ cannot be true



The refutation rule

Suppose you want to prove φ .

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction \perp ,
- then $\neg\varphi$ cannot be true
- so you actually have φ .



Proving the axioms (3)

The axiom

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can be proved from *modus ponens*, *deduction* and *refutation*:

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2	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> ψ </td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> <table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> $\neg\varphi$ </td> </tr> </table> </td> </tr> </table>	ψ	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"> $\neg\varphi$ </td> </tr> </table>	$\neg\varphi$
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modus ponens 3,1

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4	<div style="border-left: 1px solid black; padding-left: 10px;"> $\neg\psi$ </div>	modus ponens 3,1
5	\perp	modus ponens 2,4

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8	$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$	deduction 1-7

Proving the axioms (3)

The axiom

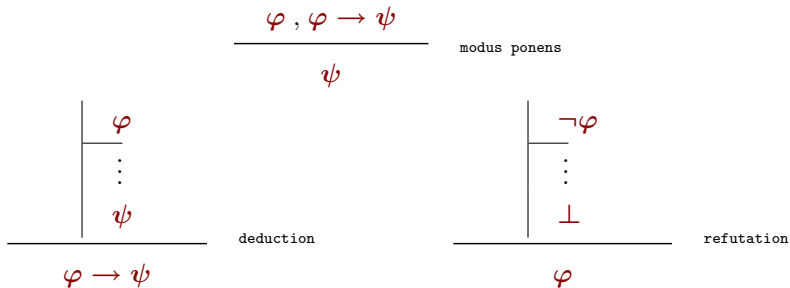
$$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

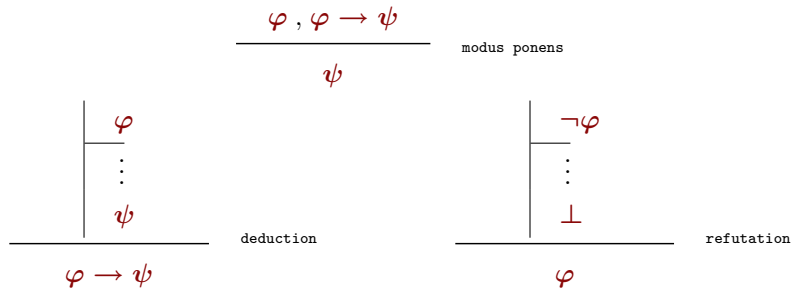
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6	φ	refutation 3-5	
7	$\psi \rightarrow \varphi$	deduction 2-6	
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For step 5, note that $\neg\psi$ can be seen as an abbreviation of $\psi \rightarrow \perp$.

So ...



So ...



The *modus ponens*, *deduction* and *refutation* rules are a complete system for propositional logic.

To facilitate things ...

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- Natural deduction introduces rules to manipulate all the connectives in an easy way.

For implication \rightarrow

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$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \text{modus ponens}$$

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E_→

For implication \rightarrow

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \text{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \quad \text{deduction}$$

E_→

For implication \rightarrow

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \text{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \quad \text{deduction}$$

E_→**I_→**

For negation \neg

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$$\frac{\neg\varphi, \varphi}{\perp}$$

For negation \neg

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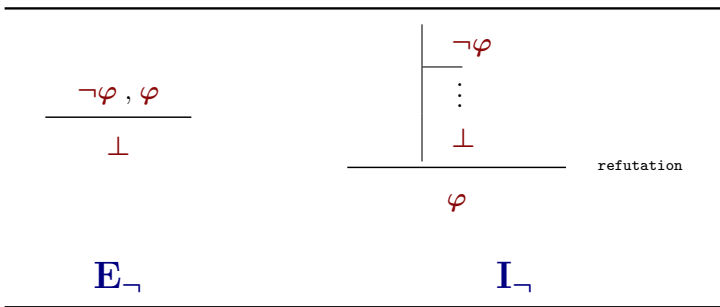
E_¬

For negation \neg

$$\frac{\neg\varphi, \varphi}{\perp}$$

$$\frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \perp \end{array}}{\varphi} \text{ refutation}$$

E_¬

For negation \neg 

For conjunction \wedge

For conjunction \wedge

$$\frac{\varphi \wedge \psi}{\varphi}$$
$$\frac{\varphi \wedge \psi}{\psi}$$

For conjunction \wedge

$$\frac{\varphi \wedge \psi}{\varphi}$$
$$\frac{\varphi \wedge \psi}{\psi}$$
$$\mathbf{E}_{\wedge}$$

For conjunction \wedge

$$\begin{array}{c}
 \hline
 \varphi \wedge \psi \\
 \hline
 \varphi
 \end{array}
 \qquad
 \begin{array}{c}
 \varphi, \psi \\
 \hline
 \varphi \wedge \psi
 \end{array}$$

$$\begin{array}{c}
 \varphi \wedge \psi \\
 \hline
 \psi
 \end{array}$$

$$\mathbf{E}_{\wedge}$$

For conjunction \wedge

$\varphi \wedge \psi$	
φ	
$\varphi \wedge \psi$	φ, ψ
ψ	$\varphi \wedge \psi$
E_∧	I_∧

For disjunction \vee

For disjunction \vee

$$\begin{array}{c}
 \varphi \vee \psi, \quad \left| \begin{array}{c} \varphi \\ \vdots \\ \chi \end{array} \right. , \quad \left| \begin{array}{c} \psi \\ \vdots \\ \chi \end{array} \right. \\
 \hline
 \chi
 \end{array}$$

For disjunction \vee

$$\frac{\varphi \vee \psi, \quad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array}, \quad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

 \mathbf{E}_{\vee}

For disjunction \vee

$$\begin{array}{c}
 \varphi \vee \psi, \quad \left| \begin{array}{c} \varphi \\ \vdots \\ \chi \end{array} \right. , \quad \left| \begin{array}{c} \psi \\ \vdots \\ \chi \end{array} \right. \\
 \hline
 \chi
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\varphi}{\varphi \vee \psi} \\
 \\
 \frac{\psi}{\varphi \vee \psi}
 \end{array}$$

 \mathbf{E}_{\vee}

For disjunction \vee

$$\begin{array}{c}
 \varphi \vee \psi, \quad \left| \begin{array}{c} \varphi \\ \vdots \\ \chi \end{array} \right. , \quad \left| \begin{array}{c} \psi \\ \vdots \\ \chi \end{array} \right. \\
 \hline
 \chi
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\varphi}{\varphi \vee \psi} \\
 \\
 \frac{\psi}{\varphi \vee \psi}
 \end{array}$$

 \mathbf{E}_{\vee} \mathbf{I}_{\vee}

For *predicate* logic

In order to present *introduction* and *elimination* rules for both \forall and \exists , we need to recall two notions.

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- **Bounded variable.**

For *predicate* logic

In order to present *introduction* and *elimination* rules for both \forall and \exists , we need to recall two notions.

- **Bounded variable.**
- **Substitution of a variable for a term in a formula.**

Bounded variable

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- **Scope of a quantifier.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the subformula φ is said to be **the scope** of the quantifier \forall (\exists).

Bounded variable

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- **Binding a variable.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the quantifier \forall (\exists) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ .

Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the subformula φ is said to be **the scope** of the quantifier \forall (\exists).
- **Binding a variable.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the quantifier \forall (\exists) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ .
- **Bounded variable.** An occurrence of a variable x is **bounded** in a formula φ if there is a quantifier in φ that binds it.

Substitution (1)

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- **Substitution inside a term.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)_t^y$$

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$$(s)_t^y$$

- Formally,

For a **constant**: $(c)_t^y := c$

For a **variable**: $\begin{cases} (x)_t^y := x & \text{for } x \text{ different from } y \\ (y)_t^y := t \end{cases}$

Substitution (1)

- **Substitution inside a term.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)_t^y$$

- Formally,

For a **constant**: $(c)_t^y := c$

For a **variable**: $\begin{cases} (x)_t^y := x & \text{for } x \text{ different from } y \\ (y)_t^y := t \end{cases}$

Examples:

$$(a)_c^x := a$$

$$(x)_a^y := x$$

$$(z)_y^z := y$$

Substitution (2)

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- **Substitution inside a formula.** Replacing the **free** occurrences of the variable y for the term t inside the **formula** φ produces the **formula** denoted by

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- **Substitution inside a formula.** Replacing the **free** occurrences of the variable y for the term t inside the **formula** φ produces the **formula** denoted by

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- Formally,

$$\begin{array}{l}
 (Pt_1 \cdots t_n)_t^y := P(t_1)_t^y \cdots (t_n)_t^y \\
 (\neg\varphi)_t^y := \neg(\varphi)_t^y \\
 (\varphi \wedge \psi)_t^y := (\varphi)_t^y \wedge (\psi)_t^y \\
 (\varphi \vee \psi)_t^y := (\varphi)_t^y \vee (\psi)_t^y \\
 (\varphi \rightarrow \psi)_t^y := (\varphi)_t^y \rightarrow (\psi)_t^y \\
 (\varphi \leftrightarrow \psi)_t^y := (\varphi)_t^y \leftrightarrow (\psi)_t^y
 \end{array}
 \left\{ \begin{array}{l}
 (\forall x\varphi)_t^y := \forall x(\varphi)_t^y \\
 (\forall y\varphi)_t^y := \forall y\varphi
 \end{array} \right.$$

$$\left\{ \begin{array}{l}
 (\exists x\varphi)_t^y := \exists x(\varphi)_t^y \\
 (\exists y\varphi)_t^y := \exists y\varphi
 \end{array} \right.$$

For the universal quantifier \forall

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$$\frac{\forall x \varphi}{(\varphi)_t^x}$$

provided that no variable in t
occurs bounded in φ

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E_{\forall}

$$\frac{\begin{array}{c} u \\ | \\ \text{---} \\ | \\ \vdots \\ | \\ (\varphi)_u^x \end{array}}{\forall x \varphi}$$

for u a special symbol not used
anywhere else in the proof

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$$\mathbf{I}_{\forall}$$

For the existential quantifier \exists 

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$$\begin{array}{c}
 \exists x \varphi, \quad \begin{array}{|l} u \\ \hline (\varphi)_u^x \\ \vdots \\ \psi \end{array} \\
 \hline
 \psi
 \end{array}$$

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 \hline
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 \end{array}$$

for u a special symbol not used
anywhere in the proof

E_∃

For the existential quantifier \exists

$$\frac{\exists x \varphi, \quad \begin{array}{c|c} u & (\varphi)_u^x \\ \hline & \vdots \\ & \psi \end{array}}{\psi}$$

for u a special symbol not used anywhere in the proof

$$\frac{(\varphi)_t^x}{\exists x \varphi}$$

provided that no variable in t occurs bounded in φ

E_∃

For the existential quantifier \exists

$$\begin{array}{c}
 \exists x \varphi, \quad \begin{array}{|l} u \\ \hline (\varphi)_u^x \\ \vdots \\ \psi \end{array} \\
 \hline
 \psi
 \end{array}$$

for u a special symbol not used
anywhere in the proof

E_{\exists}

$$\frac{(\varphi)_t^x}{\exists x \varphi}$$

provided that no variable in t
occurs bounded in φ

I_{\exists}

For the identity symbol =

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$$\frac{t_1 = t_2, \varphi}{\varphi[t_1/t_2]}$$

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where $\varphi[t_1/t_2]$ is the result of replacing, in φ , some occurrences of t_2 by t_1 , provided that

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- t_1 contains only variables that do not get bounded after replacement.

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$$\mathbf{E}_=$$

For the identity symbol $=$

$$\begin{array}{c}
 \frac{t_1 = t_2, \varphi}{\varphi[t_1/t_2]} \\
 \frac{t_1 = t_2, \varphi}{\varphi[t_2/t_1]}
 \end{array}
 \qquad
 \frac{}{t = t}$$

where $\varphi[t_1/t_2]$ is the result of replacing, in φ , some occurrences of t_2 by t_1 , provided that

- t_2 contains only variables that occur freely in φ , and
- t_1 contains only variables that do not get bounded after replacement.

for any term t .

E₌

For the identity symbol =

$$\begin{array}{c}
 \frac{t_1 = t_2, \varphi}{\varphi[t_1/t_2]} \\
 \frac{t_1 = t_2, \varphi}{\varphi[t_2/t_1]}
 \end{array}
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where $\varphi[t_1/t_2]$ is the result of replacing, in φ , some occurrences of t_2 by t_1 , provided that

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$$\mathbf{E}_=$$

$$\mathbf{I}_=$$